Gluing constructions amongst constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$

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Abstract Four constructions of constant mean curvature (CMC) hypersurfaces in $\mathbb{S}^{n+1}$ are given, which should be considered analogues of 'classical' constructions that are possible for CMC hypersurfaces in Euclidean space. First, Delaunay-like hypersurfaces, consisting roughly of a chain of hyperspheres winding multiple times around an equator, are shown to exist for all the values of the mean curvature. Second, a hypersurface is constructed which consists of two chains of spheres winding around a pair of orthogonal equators, showing that Delaunay-like hypersurfaces can be fused together in a symmetric manner. Third, a Delaunay-like handle can be attached to a generalized Clifford torus of the same mean curvature. Finally, two generalized Clifford tori of equal but opposite mean curvature of any magnitude can be attached to each other by symmetrically positioned Delaunay-like 'arms'. This last result extends Butscher and Pacard's doubling construction for generalized Clifford tori of small mean curvature.

1 Introduction

1.1 Gluing constructions of CMC hypersurfaces

A CMC hypersurface Λ contained in an ambient Riemannian manifold X of dimension $n + 1$ has the property that its mean curvature with respect to the induced metric is constant. This property ensures that n-dimensional area of Λ is a critical value of the area functional for hypersurfaces of X subject to an enclosed-volume constraint. The CMC hypersurfaces have been the objects of great interest since the beginning of modern differential geometry. Classical examples of non-trivial CMC surfaces in three-dimensional Euclidean space $\mathbb{R}^3$

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are the sphere, the cylinder and the Delaunay surfaces, and for a long while these were the only known CMC surfaces. In fact, a result of Alexandrov [2] states that the only compact, connected, embedded CMC surfaces in $\mathbb{R}^3$ are spheres.

The theory of CMC surfaces in $\mathbb{R}^3$ took a leap forward with Wente's discovery of a family of compact, though immersed, CMC tori in 1986 [29], and has progressed considerably since then. For instance, the techniques used by Wente have culminated in a representation for CMC surfaces in $\mathbb{R}^3$ akin to the classically known Weierstraß representation of minimal surfaces in which a harmonic but not anti-conformal map from a Riemann surface to the unit sphere becomes the Gauß map of a CMC immersion into $\mathbb{R}^3$ from which the immersion can be determined [8,16]. Most important for this article, however, has been the development of the *gluing technique* for constructing CMC hypersurfaces in $\mathbb{R}^3$ from simple building blocks. This technique was pioneered by Kapouleas and uses perturbation arguments from the theory of geometric partial differential equations to construct many new CMC surfaces: e.g., compact, immersed, genus two surfaces by fusing Wente tori together [13]; as well as compact surfaces of higher genus and non-compact surfaces with arbitrary numbers of ends by fusing together spheres and Delaunay surfaces [14,15]. Kapouleas' discoveries have since been complemented by much research into gluing and other constructions from geometric analysis that can be performed in the class of CMC surfaces, most notably in the work of Mazzeo, Pacard and others [11,20–22], where such constructions as the attachment of Delaunay ends to CMC surfaces and other types of constructions are realized.

The corresponding picture amongst CMC hypersurfaces of higher dimension or in other ambient manifolds is not nearly as rich as it is in $\mathbb{R}^3$. There is a certain amount of literature on CMC hypersurfaces in hyperbolic space [4,23,26,27]; but due to the non-compactness of hyperbolic space, this theory can be considered not such a vast departure from the theory of CMC hypersurfaces in $\mathbb{R}^{n+1}$. Much less is known when the ambient space is the sphere. The classically known examples in $\mathbb{S}^{n+1}$ are the hyperspheres obtained from intersecting $\mathbb{S}^{n+1}$ with affine hyperplanes, and the so-called generalized Clifford tori which are products of lower-dimensional spheres of the form $\mathbb{T}_\alpha^{p,q} := \mathbb{S}^p\big(\cos(\alpha)\big) \times \mathbb{S}^q\big(\sin(\alpha)\big)$ for $p + q = n$ and $\alpha \in (0, \pi/2)$. These are embedded hypersurfaces in $\mathbb{S}^{n+1}$ with CMC equal to $H_\alpha := q \cot(\alpha) - p \tan(\alpha)$. There are few other examples; and up to now, no truly general methods for the construction of CMC surfaces in $\mathbb{S}^{n+1}$ have been developed. However, the gluing technique is well-adapted to generalization to other settings, for philosophical as well as practical reasons: gluing is meant to assemble complex objects from simple ones, and this principle should hold equally well in $\mathbb{S}^{n+1}$ as in $\mathbb{R}^{n+1}$ because one has simple building blocks in $\mathbb{S}^{n+1}$ and a large group of isometries with which to move them around; and many of the operations involved in a gluing construction—such as forming connected sums using small bridging surfaces near a point of mutual tangency—are all *local* and thus have straightforward generalizations to $\mathbb{S}^{n+1}$.

This article adapts the gluing technique for constructing CMC hypersurfaces in $\mathbb{S}^{n+1}$ and uses it to realize various analogues of the 'classical' constructions that are possible in Euclidean space. The central idea is to position hyperspheres and/or generalized Clifford tori of the same mean curvature throughout $\mathbb{S}^{n+1}$ in various ways such that each building blocks is separated from its neighbours by a small amount. After appropriate small modifications of this initial configuration, catenoidal necks are then inserted between the building blocks at the points where they come closest to each other. This *approximate solution* of the gluing problem now has approximately CMC, where the error between its mean curvature and the mean curvature of the original building blocks is small, except

in the neck regions where the approximate solution is almost singular. One then attempts to perturb the approximate solution by a small normal deformation until it has exactly CMC.

As is often the case in gluing constructions, there are obstructions to the solvability of the CMC equation that come from the *Jacobi fields* of the approximate solution. These are eigenfunctions of the linearized mean curvature operator with zero or small eigenvalue tending to zero as the approximate solution becomes singular. As is well known, their origin is geometric: every one-parameter family of isometries of the approximate solution generates a Jacobi field with zero eigenvalue; and every one parameter family of isometries, modified by a cut-off function so that it only moves one of the building blocks of the approximate solution while fixing all the others, generates a space of functions which approximates the small eigenspaces.

The key to overcoming these obstructions and realizing the CMC gluing construction in a particular case is to combine two techniques: *symmetry conditions* and *force balancing arguments*. Both of these techniques play an important role in the gluing constructions in Euclidean space and continue to do so here, though it turns out that the symmetries of the initial configurations play a more prominent role here than they do in Euclidean space.

The role of the symmetry conditions is as follows. If the approximate solution is invariant under a group of isometries of $\mathbb{S}^{n+1}$, then one can attempt to perturb the approximate solution in an equivariant manner. Any Jacobi fields that are *not* invariant under the isometry group are thus automatically excluded from consideration. If it turns out that the isometry group is sufficiently large so that *all* Jacobi fields are excluded, then the obstructions to the solvability of the CMC equation disappear and the perturbation to an exactly CMC hypersurface can be realized.

However, if it turns out that the approximate solution possesses Jacobi fields that are invariant under all of its isometries, then the method above can not succeed in eliminating all the obstructions to the solvability of the CMC equations. This is where the so-called force balancing arguments come into play. The way to proceed is to derive the *balancing condition* that must be satisfied by the approximate solution if a perturbation to an exactly CMC hypersurface is to exist. The idea is that the obstructions provided by the invariant Jacobi fields can be avoided exactly when the approximate solution satisfies this balancing condition.

The balancing condition is best explained in the more general context found in Korevaar et al.'s work [17]. First, suppose that Λ is a hypersurface with CMC, h, in an $(n+1)$-dimensional Riemannian manifold (X, g) possessing a Killing field V. Let $\mathcal{U}$ be an open set in Λ and $\bar{\mathcal{U}}$ be an open set in X such that $\partial\bar{\mathcal{U}} = \partial\mathcal{U} \cup Q$ where Q is a bounded n-dimensional hypersurface. Then the first variation formula for the n-volume of $\mathcal{U}$ subject to the constraint of constant enclosed $(n+1)$-volume of $\bar{\mathcal{U}}$ in the direction of the variation determined by V implies

$$\int_{\partial\mathcal{U}} g(\nu, V) - h \int_{Q} g(N, V) = 0 \tag{1}$$

where ν is the unit normal vector field of $\partial\mathcal{U}$ in Λ and N is the unit normal vector field of Q in X. One can now apply this formula to the approximate solution of the CMC perturbation problem, having mean curvature approximately equal to h, in the following way. Choose the open set $\mathcal{U}$ as one of the building blocks of the approximate solution. Then $\partial\mathcal{U}$ consists of a disjoint union of small $(n-1)$-spheres at the centres of the necks attaching $\mathcal{U}$ to its neighbours, and Q is the disjoint union of the small disks that cap these spheres off.

The left hand side of (1) now encodes information about the width and location of the neck regions of $\mathcal{U}$. If the left hand side of (1) is sufficiently close to zero, then one says that $\mathcal{U}$ is in *balanced position*. The idea is now that in order to be able to overcome the obstructions to the solvability of the CMC equations, the approximate solution must be constructed in such a way that all its building blocks are in balanced position.

It is important to realize that the balancing condition amounts to a form of local symmetry satisfied by each building block with respect to its nearest neighbours. This is similar to what happens in Euclidean space. However, because compact hypersurfaces of the sphere must close up on themselves, it turns out that certain global symmetries are a consequence of the totality of local symmetries imposed by the balancing condition. It is for this reason that symmetry conditions play a more prominent role in the determination of initial configurations of building blocks that can be glued together and perturbed to have CMC.

Force balancing in itself is not the end of the story; and a balanced approximate solution can not necessarily be perturbed to an exactly CMC hypersurface. It is in addition necessary to be able to re-position the various building blocks with respect to each other so as to maintain the force balancing condition even if these are deformed by a small perturbation. Technically speaking, this amounts to the requirement that the mapping taking a re-positioned approximate solution to a set of small real numbers via the integrals on the left hand side of (1) be invertible. This requirement is well known and makes an appearance in the CMC gluing problem in Euclidean space. It also arises here, and is in fact more difficult to achieve than in Euclidean space. However, the imposition of additional symmetry conditions on the initial configuration of building blocks is enough to overcome this problem.

1.2 Statement of results

The following analogues in $\mathbb{S}^{n+1}$ of 'classical' constructions amongst CMC hypersurfaces in Euclidean space will be realized in this article. First, hypersurfaces will be constructed which consist of a chain of equally spaced hyperspheres winding a number of times around an equator. These should be construed as analogues of the Delaunay hypersurfaces, and will be called *Delaunay-like*. Depending on the choices made in the construction, these can be embedded, immersed or non-compact. Next, one can attempt to fuse a number of Delaunay-like hypersurfaces together. This fusion will be realized in one particular instance, thereby illustrating the method that could be used for more ambitious situations. Next, a more general problem along the lines of the fusion of two Delaunay-like hypersurfaces is adding Delaunay-like handles to a given CMC hypersurface. This construction will be realized again in one particular instance, where the CMC hypersurface in question is a generalized Clifford torus. Finally, it is possible to *double* two generalized Clifford tori having equal but opposite mean curvature of any magnitude by attaching them to each other with a number of symmetrically placed Delaunay-like 'arms'. This is not so much the analogue of a 'classical' construction amongst CMC hypersurfaces but rather an extension of the result obtained by Butscher and Pacard in [5,6] in which two generalized Clifford tori of equal but opposite mean curvature very close to zero are doubled by attaching them to each other with a number of symmetrically placed small catenoidal necks.

The spirit of this article is to give a collection of examples rather than fully general constructions. The reasons for this are two-fold. First, it is beneficial for one's intuition to have readily visualizable concrete examples of CMC hypersurfaces of $\mathbb{S}^{n+1}$ at hand. Second, the details of the construction vary greatly depending upon the symmetry group of the initial configuration, the invariant Jacobi fields of the approximate solution, the number of windings

needed for a configuration to close up and so on. Therefore it is best to illustrate the methods used in the gluing construction in a few special cases, which can then be readily adapted to other settings.

1.2.1 Delaunay-like CMC hypersurfaces of $\mathbb{S}^{n+1}$

Let S_α be the hypersphere obtained by intersecting $\mathbb{S}^{n+1}$ by an affine hyperplane passing a distance $\cos(\alpha) \in (0, 1)$ from the origin. The mean curvature of S_α is the constant $H_\alpha := n \cot(\alpha)$.

Main Theorem 1 *Suppose γ is the great circle in $\mathbb{S}^{n+1}$ generated by the one-parameter family of rotations $R_\theta \in SO(n + 2)$. For every $\alpha \in (0, \pi/2)$ and sufficiently small $\tau > 0$ there exists a CMC hypersurface $\Lambda_{\alpha,\tau}$ of mean curvature $H_\alpha := n \cot(\alpha)$ which is approximately equal to a union of hyperspheres of the form $R_\theta^k(S_\alpha)$ that are separated by a distance τ from each other and connected by small catenoidal necks. These hypersurfaces are either non-compact and immersed, compact and immersed, or compact and embedded depending on the values of α and τ. Finally, as $\tau \to 0$, then $\Lambda_{\alpha,\tau}$ converges in the C^∞ topology to the union of hyperspheres of mean curvature H_α positioned end-to-end along γ.*

It should be noted that the result above can be viewed in a different and a priori simpler light. That is to say, one first considers $\mathbb{S}^{n+1} \subseteq \mathbb{R}^2 \times \mathbb{R}^n$ and allows the subgroup $O(n) \subseteq O(n + 2)$ to act on $\mathbb{S}^{n+1}$ by its action on the last n coordinates. Then one asks for CMC surfaces of $\mathbb{S}^n$ that are invariant by the action of $O(n)$, and the CMC condition reduces to an ODE in this *Ansatz*. The resulting hypersurfaces are the so-called *rotational* CMC hypersurfaces of $\mathbb{S}^n$ and the Delaunay-like hypersurfaces $\Lambda_{\alpha,\tau}$ are examples of such surfaces. The ODEs associated to rotational hypersurfaces satisfying various geometric conditions have been studied by several authors, including [3,7,19,25,28]. Though the specific question addressed here—of the existence of compact, embedded rotational CMC hypersurfaces akin to Delaunay hypersurfaces—has not been tackled directly by any of these authors, it is fairly clear that the ODE techniques deployed by these authors could be used to prove a result such as Main Theorem 1. Furthermore, there is numerical evidence to support the existence of these hypersurfaces—see for instance the pictures posted at the famous Geometry Analysis Numerics and Graphics website www.gang.umass.edu. But a complete theoretical picture of these hypersurfaces is still lacking. For instance, an ODE analysis of the type found in [10, Sect. 2] must still be done in order to obtain the kind of precision offered by Main Theorem 1.

Since the PDE techniques of Main Theorem 1 do yield a very precise description of the solutions of the ODE for rotational CMC hypersurfaces, at least those with small neck size, and since Main Theorem 1 provides a simple context in which to present the techniques which will be re-used to prove the three remaining Main Theorems of this article, the proof of Main Theorem 1 will be given in complete detail.

1.2.2 Gluing two Delaunay-like hypersurfaces together

One can imagine a large variety of ways in which two or more Delaunay-like hypersurfaces can be pasted together. One simple way is to glue a Delaunay-like hypersurface to one whose equator has been rotated. The resulting configuration consists of two sets of hyperspheres, winding around two different equators. The only overlap occurs at two antipodal hyperspheres; this is where the first Delaunay-like hypersurface will be glued to the second. It is

worth noting that every such configuration is in balanced position—this is because balancing requires only that when a hypersphere is to be glued to its neighbours at either two or four points, then these points are opposite each other. Consequently, the balancing map can not be invertible as a function of the geometry of the initial configuration and the CMC perturbation can not be realized. This reflects the fact that one does *not* expect to find families of solutions using the gluing technique in a compact setting. However, if the angle made by the equators is $\pi/2$ then the initial configuration is invariant under additional reflection symmetries. Now force balancing arguments together with symmetry conditions make it possible to realize the CMC perturbation.

Main Theorem 2 *Let $\Lambda_{\alpha,\tau}$ denote the Delaunay-like CMC hypersurface winding around the geodesic γ constructed in Main Theorem 1. Let $Q \in SO(n+2)$ be a rotation by an angle of $\pi/2$ about an axis perpendicular to the normal vector that defines γ. Then for suitable choices of α and τ, it is possible to form the fusion of $\Lambda_{\alpha,\tau}$ and $Q(\Lambda_{\alpha,\tau})$ as described above. Moreover, if both of these hypersurfaces are embedded, then so is their fusion.*

One could extend Main Theorem 2 by fusing $\Lambda_{\alpha,\tau}$ to k Delaunay surfaces of the form $R_k \circ Q(\Lambda_{\alpha,\tau})$ where $R_k \in SO(n+2)$ is a rotation preserving $\Lambda_{\alpha,\tau}$ and satisfying $R_k^k = Id$ and commuting with the reflection symmetry that reverses the direction of γ. This would be a more or less straightforward extension of the ideas used in the proof of Main Theorem 2.

1.2.3 Attaching Delaunay-like handles to Clifford tori

Similar methods as in the proof of Main Theorem 2 can be used to add Delaunay-like *handles* to a CMC hypersurface of $\mathbb{S}^{n+1}$ under certain circumstances. That is, suppose Λ is a CMC hypersurface in $\mathbb{S}^{n+1}$ of CMC H such that there are two points p and q in Λ that can be connected to each other by a geodesic γ that ideally does not touch Λ in any other point. Then if the initial configuration consisting of Λ and hyperspheres of parameter $\alpha = \mathrm{arccot}(H/n)$ positioned along γ possesses enough symmetry, then one might expect that the perturbation and force-balancing arguments of Main Theorem 2 carry through.

The symmetry group of Λ as well as its Jacobi fields clearly play a crucial role in determining if the construction outlined above is possible. Instead of stating general conditions on an arbitrary CMC hypersurface Λ under which the perturbation to a CMC hypersurface can be executed, the third main theorem of this article is to give one concrete example of the construction which can then serve as an archetype for more general constructions. In this example, Λ is the generalized Clifford torus $\mathbb{T}_{\alpha}^{n_1,n_2} := \{(\cos(\alpha)\Theta_1, \sin(\alpha)\Theta_2) : \Theta_j \in \mathbb{S}^{n_j}\} \subseteq \mathbb{R}^{n_1+1} \times \mathbb{R}^{n_2+1}$ where $n_1 + n_2 = n$, which has mean curvature $H_\alpha := n_2 \cot(\alpha) - n_1 \tan(\alpha)$.

Main Theorem 3 *Let $\mathbb{T}_{\alpha}^{n_1,n_2}$ be the generalized Clifford torus of $\mathbb{S}^{n+1}$. Then there is a geodesic γ that meets $\mathbb{T}_{\alpha}^{n_1,n_2}$ orthogonally in two points p, p'. For suitable choices of $\alpha \in (0, \pi/2)$, small τ and positive integers N and m, there exists a CMC hypersurface of $\mathbb{S}^{n+1}$ which is approximately equal to the connected sum at the points p, p' of $\mathbb{T}_{\alpha}^{n_1,n_2}$ and a truncated Delaunay-like hypersurface consisting of N spherical regions separated a distance τ from each other and winding m times around γ. If α is sufficiently close to $\pi/2$ then it is possible to have $m = 1$ in this construction, and the resulting hypersurface is embedded.*

It should be possible to extend Main Theorem 3 by attaching multiple handles to $\mathbb{T}_{\alpha}^{n_1,n_2}$, located at a collection p_k, p_k' of attachment points. A symmetry condition on the locations of these points will then govern the existence of the CMC hypersurface with multiple handles.

This symmetry condition will be akin to the symmetry condition found in Main Theorem 4 below.

1.2.4 Doubling Clifford tori—revisited

Butscher and Pacard [5,6] have constructed in CMC hypersurfaces of small but non-zero mean curvature by *doubling* the unique minimal Clifford torus in the family of Clifford tori of $\mathbb{S}^{n+1}$. That is, these hypersurfaces are small perturbations of two nearby parallel translates of $\mathbb{T}_{\alpha_*}^{n_1,n_2}$, where $\alpha_* = \arctan(\sqrt{n_2/n_1})$, which are glued together at a number of symmetrically located points $\{p_1, \ldots, p_J\}$ by means of small catenoidal necks. The conditions for the existence of these hypersurfaces are phrased in terms of the location of the gluing points: if the gluing points are such that the symmetries of $\mathbb{T}_{\alpha_*}^{n_1,n_2} \setminus \{p_1, \ldots, p_J\}$ rule out all the Jacobi fields of the linearized mean curvature operator, then it is possible to solve the CMC equations.

The methods developed in this article allow the above construction to be extended in the following way. One can take two Clifford tori with equal but opposite mean curvature of *any* magnitude, choose a collection of gluing points on each one according to the same requirements as in [5,6], and place a sequence of hyperspheres of the same mean curvature end-to-end along the geodesics connecting the gluing points on one Clifford torus to those on the other. The entire configuration can then be glued together by inserting catenoidal necks between the hyperspheres, and then perturbed to have exactly CMC. The symmetry condition on the location of the points $\{p_1, \ldots, p_J\}$ is exactly the same as the one used in [5,6]. That is, if the full group of symmetries preserving $\mathbb{T}_{\alpha}^{n_1,n_2}$ is $O(n_1+1) \times O(n_2+1) \subseteq O(n+2)$ acting diagonally on $\mathbb{R}^{n+2} = \mathbb{R}^{n_1+1} \times \mathbb{R}^{n_2+1}$, then the finite subgroups of interest will be of the form $G \subseteq \{(\omega^1, \omega^2) : \omega^j \in O(n_j+1)\}$ and will contain the element $T := (T^1, T^2)$ where $T^j \in O(n_j+1)$ is the reflection symmetry across the x_j^1-axis given by $T^j\left(x_j^1, x_j^2, \ldots, x_j^{n_j+1}\right) := \left(x_j^1, -x_j^2, \ldots, -x_j^{n_j+1}\right)$.

Main Theorem 4 *Consider two generalized Clifford tori $\mathbb{T}_{\alpha}^{n_1,n_2}$ and $\mathbb{T}_{\bar{\alpha}}^{n_1,n_2}$ with equal but opposite mean curvature $H_{\alpha} = -H_{\bar{\alpha}}$ and a finite subgroup $G \subseteq O(n+2)$ of the type discussed above. Assume that there are no G-invariant functions of the form*

$$(x_1, x_2) \longmapsto \sum_{j=1}^{n_1+1} \sum_{j'=1}^{n_2+1} a_{jj'} x_1^j x_2^{j'}$$

where $a_{jj'} \in \mathbb{R}$. Let p and $\bar{p}$ be two points in $\mathbb{T}_{\alpha}^{n_1,n_2}$ and $\mathbb{T}_{\bar{\alpha}}^{n_1,n_2}$, respectively, that are connected to each other by the normal geodesic γ. Let $M = \{p_1 = p, \ldots, p_{|G|}\}$ and $\bar{M} = \{\bar{p}_1 = \bar{p}, \ldots, \bar{p}_{|G|}\}$ be the orbits under G of p and $\bar{p}$, and let γ_s be the geodesic separating p_s from $\bar{p}_s$. For suitable choices of $\alpha \in (0, \pi/2)$, small τ and positive integers N and m there is a G-invariant CMC hypersurface consisting roughly of $\mathbb{T}_{\alpha}^{n_1,n_2}$ and $\mathbb{T}_{\bar{\alpha}}^{n_1,n_2}$ attached to each other by a chain of N hyperspheres of the same mean curvature winding m times around each of the geodesics $\gamma_1, \ldots, \gamma_{|G|}$. For sufficiently large values of the mean curvature, it is possible to find embedded hypersurfaces of this kind.

As a final comment before launching into the proof of the first main theorem, it is important to realize what can and can not be achieved in the constructions above. The gluing technique is by its very nature a singular perturbation method. Thus the only hypersurfaces that can be created with it are nearly singular, resembling CMC building blocks separated

by an extremely small amount τ and connected by extremely small necks of high principal curvatures. Two geometric consequences of this fact are worth mentioning; and both revolve around the need to create *compact* glued CMC hypersurfaces. First, small τ and fixed mean curvature H can result in extremely large windings of the geodesics used in the constructions. The number of windings is dictated by the need for the glued configuration to close and remain compact. Moreover, a small change in τ necessarily results in a large change in the number of hyperspheres positioned along the geodesic for the same reason. Second, the only way small changes in τ can produce equally small changes in the glued configuration is if the mean curvature is allowed to change. This is because changing τ creates changes the amount of space between the building blocks. Maintaining compactness and the number of windings forces a commensurate change in the radius parameters of the building blocks, and this amounts to a change in the overall mean curvature.

2 Delaunay-like CMC hypersurfaces in $\mathbb{S}^{n+1}$

2.1 Introduction

The Delaunay surfaces in $\mathbb{R}^3$ are the classical examples of non-compact, periodic, axially symmetry CMC surfaces. They have been studied carefully by many authors, for example, [12,17,21]. In general, one can say that the Delaunay surfaces look roughly like a chain of spherical lobes connected to each other by neck regions. There are two parameters that naturally describe the Delaunay surfaces, namely, the mean curvature and the width of the neck at its narrowest point. (This latter parameter is related in a complicated manner to the period of the Delaunay surface.) There is a maximal value of the width parameter for a given mean curvature; and in this limit, the Delaunay surface is just a cylinder. When the width parameter is small, the spherical lobes approximate true spheres and the neck regions approximate truncated, re-scaled catenoids. In the limit when the width parameter approaches zero, the Delaunay surface approaches a union of osculating spheres whose points of contact are on a common line.

The geometric picture of the Delaunay surfaces in $\mathbb{R}^3$ leads to the following idea. One can attempt to construct "Delaunay-like" CMC surfaces in $\mathbb{S}^3$, at least in the small-neck-width regime, by positioning mutually tangent hyperspheres along an equator, connecting them together using truncated and re-scaled catenoidal necks, and then perturbing the resulting surface until it has CMC. This construction will be realized in this section of this article in the more general context of CMC hypersurfaces in $\mathbb{S}^{n+1}$ for arbitrary $n \geq 2$.

2.2 The building blocks of the construction

2.2.1 The hyperspheres in $\mathbb{S}^{n+1}$

The CMC hyperspheres inside $\mathbb{S}^{n+1}$ are obtained by intersecting $\mathbb{S}^{n+1}$ with affine hyperplanes. Without loss of generality, suppose that $\mathbb{R}^{n+2}$ is written as $\mathbb{R} \times \mathbb{R}^{n+1}$ and given the coordinates $(x^0, x^1, \ldots, x^{n+1})$, while the hyperplane in question is $\Pi_a := \{x \in \mathbb{R}^{n+2} : x^0 = a\}$ for some $a \in [0, 1)$. For convenience, introduce an angle $\alpha \in (0, \pi/2)$ so that $\cos(\alpha) = a$.

Definition 1 Up to $SO(n+2)$-rotation, the *affine hypersphere* with parameter $\alpha \in (0, \pi/2]$ in $\mathbb{S}^{n+1}$ is the hypersurface $S_\alpha := \mathbb{S}^{n+1} \cap \Pi_a$. In other words,

$$S_\alpha = \{x \in \mathbb{R}^{n+2} : x^0 = \cos\alpha \quad \text{and} \quad (x^1)^2 + \cdots + (x^{n+1})^2 = \sin^2(\alpha)\}.$$

It is essential for the constructions that follow to have a thorough understanding of the geometry of *normal graphs* over S_α. Let $\mathbf{\Theta} : \mathbb{S}^n \to \mathbb{R}^{n+1}$ be a parametrization of the unit sphere in $\mathbb{R}^{n+1}$. Then one can parametrize S_α via $\mathbf{\Theta} \longmapsto (\cos(\alpha), \sin(\alpha)\mathbf{\Theta})$. Furthermore, the displacement by a distance σ along the geodesic normal to a point on S_α is found using the exponential map and is given by

$$\exp(\sigma N_\alpha)(\mathbf{\Theta}) = (\cos(\alpha + \sigma), \sin(\alpha + \sigma)\mathbf{\Theta})$$

where N_α is the unit outward normal of S_α, given explicitly by $N_\alpha := -\cot(\alpha)P + \sin(\alpha)\frac{\partial}{\partial x^0}$ and $P := \sum_{k=1}^{n+1} x^k \frac{\partial}{\partial x^k}$ is the position vector field of the $\mathbb{R}^{n+1}$ factor. Suppose now that $F : \mathbb{S}^n \to \mathbb{R}$ is a function on $\mathbb{S}^n$. Then one can parametrize the normal graph over S_α corresponding to F via

$$\mathbf{\Theta} \longmapsto (\cos(\alpha + F(\mathbf{\Theta})), \sin(\alpha + F(\mathbf{\Theta}))\mathbf{\Theta}).$$

The geometric features of normal graphs over S_α that will be relevant later on are collected in the following lemma.

Lemma 2 *The following facts hold for the normal graph over S_α corresponding to the function $F : \mathbb{S}^n \to \mathbb{R}$. The norms and derivatives on the right hand sides of the expressions below are those of the standard metric on $\mathbb{S}^n$.*

1. *The induced metric is*

$$g = dF \otimes dF + \sin^2(\alpha + F)g_{\mathbb{S}^n}.$$

2. *The Laplacian of g satisfies*

$$\sin^2(\alpha + F)\Delta_g(u) = \Delta u - \frac{\nabla^2 u(\nabla F, \nabla F)}{A^2}$$
$$+ \frac{((n-2)\sin(\alpha + F)\cos(\alpha + F) - \Delta F)\nabla F \cdot \nabla u}{A^2}$$
$$+ \frac{(\nabla^2 F(\nabla F, \nabla F) + \sin(\alpha + F)\cos(\alpha + F)\|\nabla F\|^2)\nabla F \cdot \nabla u}{A^4}$$

   where $A := (\sin^2(\alpha + F) + \|\nabla F\|^2)^{1/2}$.
3. *The unit normal vector field N_α is chosen to be*

$$AN_\alpha := \sin^2(\alpha + F)\frac{\partial}{\partial x^0} - \sum_{j=1}^{n+2}\left(\cos(\alpha + F)\sin(\alpha + F)\mathbf{\Theta}^j - [g_{\mathbb{S}^n}]^{st}\nabla_s F\frac{\partial \mathbf{\Theta}^j}{\partial v^t}\right)\frac{\partial}{\partial x^j}$$

   where $v \longmapsto \Theta(v)$ parametrizes the unit n-sphere.
4. *The second fundamental form satisfies*

$$AB_{st} = -\sin(\alpha + F)\nabla_s\nabla_t F + 2\cos(\alpha + F)\nabla_s F \nabla_t F$$
$$+ \sin^2(\alpha + F)\cos(\alpha + F)[g_{\mathbb{S}^n}]_{st}.$$

5. *The mean curvature satisfies*

$$A\sin(\alpha + F)H = -\Delta F + n\sin(\alpha + F)\cos(\alpha + F)$$
$$- \frac{\nabla^2 F(\nabla F, \nabla F) - \cos(\alpha + F)\sin(\alpha + F)\|\nabla F\|^2}{A^2}.$$

Note that when $F \equiv 0$ is substituted into the various quantities in Lemma 2, one obtains the corresponding quantities for S_α itself. That is, the induced metric is $g_\alpha := \sin^2(\alpha)g_{\mathbb{S}^n}$ where $g_{\mathbb{S}^n}$ is the standard metric of the n-sphere, the second fundamental form is $B_\alpha := \cos(\alpha)\sin(\alpha)g_{\mathbb{S}^n}$ and mean curvature is $H_\alpha := n\cot(\alpha)$. Therefore one sees that the second fundamental form of S_α is parallel and has CMC.

2.2.2 The generalized catenoid in $\mathbb{R}^{n+1}$

The generalized catenoid in $\mathbb{R}^{n+1}$ is the n-dimensional analogue of the standard catenoid, namely, the unique, cylindrically symmetric, minimal hypersurface in $\mathbb{R}^{n+1}$. The derivation of the generalized catenoid (hereinafter referred to as simply the catenoid) is a fairly simple exercise in Riemannian geometry and will not be carried out here. Since the zero mean curvature condition is invariant under scaling, there is a one-parameter family of such minimal hypersurfaces. The definition of the scaled catenoid is as follows, in which $\mathbb{R}^{n+1}$ is written $\mathbb{R} \times \mathbb{R}^n$ using coordinates $(y^1, \dots, y^{n+1})$ with $\hat{y} := (y^2, \dots, y^{n+1})$.

Definition 3 The ε-scaled catenoid in $\mathbb{R} \times \mathbb{R}^n$ is the hypersurface $\varepsilon\Sigma$ parameterized by

$$(s, \Theta) \in \mathbb{R} \times \mathbb{S}^{n-1} \longmapsto \varepsilon(\psi(s), \phi(s)\Theta)$$

where $\phi(s) := (\cosh(n-1)s)^{1/(n-1)}$ and $\psi(s) := \int_0^s \phi^{2-n}(\sigma)\mathrm{d}\sigma$.

An alternate parametrization of the ε-scaled catenoid will also be used in the sequel, namely, when $\varepsilon\Sigma$ is written as the union of two graphs over the $\mathbb{R}^n$ factor. That is, by inverting the equation $\|\hat{y}\| = \varepsilon\phi(s)$, one finds that $\varepsilon\Sigma = \Sigma_\varepsilon^+ \cup \Sigma_\varepsilon^-$ where $\Sigma_\varepsilon^\pm := \{(\pm F_\varepsilon(\|\hat{y}\|), \hat{y}) : \|\hat{y}\| \geq \varepsilon\}$. The function $F_\varepsilon : \{x \in \mathbb{R} : x \geq \varepsilon\} \to \mathbb{R}$ is defined by $F_\varepsilon(x) = \varepsilon F(x/\varepsilon)$ where

$$F(x) := \int_1^x (\sigma^{2n-2} - 1)^{-1/2}\mathrm{d}\sigma. \tag{2}$$

Note that in dimension $n = 2$ this function is simply $F(x) = \mathrm{arccosh}(x)$.

The geometric features of the ε-scaled catenoid that will be relevant later on are as follows. The induced metric of Σ is $g_{\varepsilon\Sigma} := \varepsilon^2\phi^2(\mathrm{d}s^2 + g_{\mathbb{S}^{n-1}})$. The unit normal vector field of Σ is chosen as

$$N_{\varepsilon\Sigma} := \frac{\dot{\phi}}{\phi}\frac{\partial}{\partial y^1} - \phi^{1-n}P_\Theta,$$

where P_Θ is the position vector field of the $\mathbb{R}^n$ factor evaluated at the point $\Theta \in \mathbb{S}^{n-1}$. Then the second fundamental form of Σ is given by

$$B_{\varepsilon\Sigma} := \varepsilon\phi^{2-n}((1-n)\mathrm{d}s^2 + g_{\mathbb{S}^{n-1}})$$

and satisfies

$$\|B_{\varepsilon\Sigma}\| = \frac{C_1}{\varepsilon\phi^n} \quad \text{and} \quad \|\nabla B_{\varepsilon\Sigma}\| = \frac{C_2 \sinh((n-1)s)}{\varepsilon^2\phi^{2n}}$$

where $C_1 = \sqrt{n(n-1)}$ and $C_2 = n\sqrt{(n+2)(n-1)}$. The mean curvature of Σ vanishes.

2.3 Assembling the approximate solution

The approximately CMC "Delaunay-like" hypersurface will be constructed roughly as follows. First consider a single hypersphere S_α with a fixed $\alpha \in (0, \pi/2)$. Introduce a small *spacing parameter* τ satisfying $0 < \tau \ll 2\pi$. Now position the rotated copies of S_α in a symmetric manner around a geodesic γ in $\mathbb{S}^{n+1}$ so that each hypersphere is separated from its two nearest neighbours by a distance τ. Depending on the choice of α and τ, either this configuration of hyperspheres winds around γ a finite number of times before repeating, or else it never does. Now insert between the points of closest approach between every pair of neighbouring hyperspheres a suitably re-scaled, truncated and embedded catenoid. The truncation and scaling parameter should be such that its height is $\mathcal{O}(\tau)$ and it makes optimally smooth contact with the hyperspheres on either side. However, for this to be possible it turns out that it is necessary also to perturb the hyperspheres very slightly to improve their 'fit' with the catenoids.

Here are the details of this construction.

2.3.1 Positioning the hyperspheres

The hyperspheres will first be positioned in the most symmetric way possible using a cyclic subgroup rotations. Define the rotation

$$R_\theta = \left(\begin{array}{cc|c} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ \hline 0 & 0 & I_n \end{array} \right)$$

where I_n is the $n \times n$ identity matrix. The rotation R_θ generates the geodesic γ formed by intersecting $\mathbb{S}^n$ with the (x^0, x^1)-plane. Straightforward trigonometry shows that any pair of rotated hyperspheres $R_{2\alpha+\tau}^k(S_\alpha)$ and $R_{2\alpha+\tau}^{k+1}(S_\alpha)$ are separated by a distance τ. The points of closest approach between these rotated hyperspheres are $R_{2\alpha+\tau}^k(p^+) \in R_{2\alpha+\tau}^k(S_\alpha)$ and $R_{2\alpha+\tau}^{k+1}(p^-) \in R_{2\alpha+\tau}^{k+1}(S_\alpha)$ where $p^\pm := R_\alpha^{\pm 1}(p) \in S_\alpha$ and p is the point $p := (1, 0, \ldots, 0)$. Furthermore, the set

$$\Lambda_{\alpha,\tau}^\# := \bigcup_{k=0}^\infty R_{2\alpha+\tau}^k(S_\alpha)$$

consists of either: a finite number of distinct rotated copies of S_α when $2\alpha + \tau$ is a rational multiple of 2π; or an infinite number of distinct rotated copies of S_α, otherwise. In the former case, if $2\alpha + \tau = 2\pi m/N$ for some pair of positive and relatively prime integers m and N, then the number of copies of S_α equals N and these N copies wind m times around γ before repeating. If $m = 1$ then they do not intersect each other at all.

2.3.2 The normal perturbation of the hyperspheres

Each hypersphere in the initial configuration must be perturbed slightly in the normal direction. In order to preserve symmetry with respect to $R_{2\alpha+\tau}$, the same perturbation will be used for each hypersphere. Thus it suffices to explain how S_α is perturbed.

This begins with the choice of a function $G : S_\alpha \to \mathbb{R}$ which determines the normal perturbation. Recall that the linearized mean curvature operator on the space of normal graphs over a hypersurface Λ in $\mathbb{S}^{n+1}$ is $\mathcal{L}_\Lambda := \Delta_\Lambda + \|B_\Lambda\|^2 + n$ where Δ_Λ is the Laplacian of Λ and B_Λ is the second fundamental form of Λ. In the case $\Lambda = S_\alpha$, then $\mathcal{L}_\alpha := \sin^{-2}(\alpha)\left(\Delta_{\mathbb{S}^n} + n\right)$.

Choose a parametrization of $S_\alpha \setminus \{p^+, p^-\}$ in which rotational symmetry around the geodesic γ is in evidence; namely, the parametrization

$$(\mu, \Theta) \in (0, \pi) \times \mathbb{S}^{n-1} \longmapsto \big(\cos(\alpha), \sin(\alpha)\cos(\mu), \sin(\alpha)\sin(\mu)\Theta\big) \tag{3}$$

where $\Theta : \mathbb{S}^{n-1} \to \mathbb{R}^n$ is a parametrization of $\mathbb{S}^{n-1}$ of the unit sphere in $\mathbb{R}^n$. Now let G be the Θ-independent solution of the equation $\mathcal{L}_\alpha(G) = 0$ which is singular at $\mu = 0$ and $\mu = \pi$ and symmetric with respect to $\mu \mapsto \pi - \mu$. Explicitly, this function is

$$G(\mu) := -\sin(\mu) - \cos(\mu) \int\limits_{\pi/2}^{\mu} \frac{1 - \sin^{n-1}(\sigma)}{\cos^2(\sigma)\sin^{n-1}(\sigma)} \, d\sigma. \tag{4}$$

Important for later on is the asymptotic expansion of G at $\mu = 0$. This is

$$G(\mu) = \begin{cases} -1 + \log 2 - \log(\mu) + \mathcal{O}(\mu^2 |\log(\mu)|) & n = 2 \\[2mm] \dfrac{1}{\mu} + \mathcal{O}(\mu) & n = 3 \\[2mm] \dfrac{1}{2\mu^2} + \mathcal{O}(1 + |\log(\mu)|) & n = 4 \\[2mm] \dfrac{1}{(n-2)\mu^{n-2}} + \mathcal{O}(\mu^{4-n}) & n \geq 5 \end{cases} \tag{5}$$

when $0 < \mu \ll \pi/2$.

The desired normal perturbation can be defined as follows. Choose a small parameter $\varepsilon > 0$ (whose relation to τ and α will be specified in due course) and define the normal graph

$$\tilde{S}_{a,\varepsilon} := \exp(\varepsilon^{n-1} G N_\alpha)(S_\alpha \setminus \{p^+, p^-\})$$

where N_α is the outward unit normal vector field of S_α. For concreteness, this hypersurface has the parametrization

$$(\mu, \Theta) \longmapsto \big(\cos(\alpha + \varepsilon^{n-1} G(\mu)), \sin(\alpha + \varepsilon^{n-1} G(\mu))\cos(\mu),$$
$$\sin(\alpha + \varepsilon^{n-1} G(\mu))\sin(\mu)\Theta\big) \tag{6}$$

as in the discussion preceding Lemma 2.

2.3.3 Stereographic coordinates adapted to a pair of hyperspheres

Canonical coordinates that are well adapted to the pair of hyperspheres $R_{2\alpha+\tau}^k(S_\alpha)$ and $R_{2\alpha+\tau}^{k+1}(S_\alpha)$ will be needed. These can be defined as follows. First, note that the point $R_{\alpha+\tau/2}^{2k+1}(p)$ lies on the geodesic γ at the midpoint between these two hyperspheres. Now let $K : \mathbb{S}^{n+1} \setminus \{-p\} \to \mathbb{R}^{n+1}$ denote the stereographic projection centred at p defined by

$$K(x^0, x^1, \ldots, x^{n+1}) := \left(\frac{x^1}{1 + x^0}, \cdots, \frac{x^{n+1}}{1 + x^0} \right).$$

Then the desired adapted coordinates are given by the inverse of the mapping $K \circ R_{\alpha+\tau/2}^{-(2k+1)}$: $\mathbb{S}^{n+1} \setminus \{-R_{\alpha+\tau/2}^{2k+1}(p)\} \to \mathbb{R}^{n+1}$.

Recall that stereographic projection sends non-equatorial k-spheres in $\mathbb{S}^{n+1}$ to k-spheres in $\mathbb{R}^{n+1}$ and sends equatorial k-spheres to linear subspaces. One thus expects that the coordinate

image of the geodesic γ is the y^1-axis. Also, one expects that the coordinate images of the two hyperspheres $R^k_{2\alpha+\tau}(S_\alpha)$ and $R^{k+1}_{2\alpha+\tau}(S_\alpha)$ are two hyperspheres symmetrically located on either side of the origin and centred at two points on the y^1-axis. Indeed, one can check that any point of $\mathbb{S}^{n+1}$ of the form (3) maps to the point $(y^1, \hat{y})$ in $\mathbb{R}^{n+1}$ given by

$$
\begin{aligned}
y^1 &= \frac{-\sin(\alpha + \tau/2)\cos(\alpha) + \cos(\alpha + \tau/2)\sin(\alpha)\cos(\mu)}{1 + \cos(\alpha + \tau/2)\cos(\alpha) + \sin(\alpha + \tau/2)\sin(\alpha)\cos(\mu)} \\
\hat{y} &= \frac{\sin(\alpha)\sin(\mu)\Theta}{1 + \cos(\alpha + \tau/2)\cos(\alpha) + \sin(\alpha + \tau/2)\sin(\alpha)\cos(\mu)}.
\end{aligned}
\tag{7}
$$

And, it is easy to check that this point lies on the locus of points satisfying the equation

$$
\left(y^1 + d\right)^2 + \|\hat{y}\|^2 = r^2
\tag{8}
$$

where

$$
\begin{aligned}
r &= r(\alpha, \tau) := \frac{\sin(\alpha)}{\cos(\alpha) + \cos(\alpha + \tau/2)} \\
d &= d(\alpha, \tau) := \frac{\sin(\alpha + \tau/2)}{\cos(\alpha) + \cos(\alpha + \tau/2)}.
\end{aligned}
\tag{9}
$$

Observe that $d - r = \tan(\tau/4)$ is displacement from the origin of the hypersphere determined by (8) in these coordinates. This yields a geometric displacement of $\tau/2$ when measured with respect to the metric $A^{-2}g_0$, as expected.

The reason for using stereographic coordinates, other than the fact that spheres map to spheres, is that these are in geodesic normal form: at the centre of the coordinates, the metric is Euclidean and its derivatives vanish. This can be seen by the computation of the metric

$$
\left(R^{2k+1}_{\alpha+\tau/2} \circ K^{-1}\right)^* g_{\mathbb{S}^{n+1}} = A^{-2}g_0
$$

where g_0 is the Euclidean metric of $\mathbb{R}^{n+1}$ and $A(y) = \frac{1}{2}\left(1 + \sum_{k=1}^{n+1}(y^k)^2\right)$ using the coordinates $(y^1, \ldots, y^{n+1})$ for $\mathbb{R}^{n+1}$. The geodesic normal form will have the effect of distorting as little as possible the geometry of objects embedded into the sphere using the stereographic coordinate map, provided one remains near the origin. The following lemma contains the results of this kind that will be used in the sequel.

Lemma 4 *Let $\sigma : \Sigma \to \mathbb{R}^{n+1}$ be an embedding of the submanifold Σ into $\mathbb{R}^{n+1}$. Let h_0, $\mathring{\Delta}$, N_0, $\mathring{B}_{ij}$ and H_0 be the induced metric, Laplacian, unit outward normal vector field, components of the second fundamental form and mean curvature of $\sigma(\Sigma)$ with respect to the Euclidean metric. The corresponding quantities with respect to the metric $g = A^{-2}g_0$ coming from stereographic projection satisfy the following identities.*

1. *The induced metric of $\sigma(\Sigma)$ with respect to g is $h = [A \circ \sigma]^{-2}h_0$. The Laplacian of $\sigma(\Sigma)$ with respect to h is*

$$
\Delta u = [A \circ \sigma]^2 \mathring{\Delta} u - \sum_{k=1}^{n+1}(n - 2)[A \circ \sigma]h_0(\sigma^k \mathring{\nabla}\sigma^k, \mathring{\nabla}u)
$$

where σ^k are the components of σ.

2. *The second fundamental form of $\phi(\Sigma)$ with respect to the metric g is*

$$
B_{ij} = [A \circ \sigma]^{-1}\mathring{B}_{ij} - [A \circ \sigma]^{-2}g_0(\sigma, N_0)[g_0]_{ij}.
$$

3. *The mean curvature of $\sigma(\Sigma)$ with respect to the metric g is*

$$H = [A \circ \sigma]H_0 - ng_0(\sigma, N_0).$$

2.3.4 Inserting truncated catenoids

The pair of perturbed hyperspheres $R^k_{2\alpha+\tau}(\tilde{S}_\alpha)$ and $R^{k+1}_{2\alpha+\tau}(\tilde{S}_\alpha)$ maps to a pair of hypersurfaces which are small deformations of $R^k_{2\alpha+\tau}(S_\alpha)$ and $R^{k+1}_{2\alpha+\tau}(S_\alpha)$. It is necessary to have a precise description of the shape of these hypersurfaces near the y^1-axis in order to determine to what extent they resemble the ends of a catenoid. This will be done by reparametrizing them as graphs over the $\hat{y}$-hyperplane in a small neighbourhood of the y^1-axis.

From Eq. 6 and the properties of the stereographic projection as per Eq. 7, one finds that the coordinates $y(\mu, \Theta) \in \mathbb{R}^{n+1}$ of a point in the stereographic projection of the hypersphere $R^k_{2\alpha+\tau}(\tilde{S}_\alpha)$ satisfy

$$y^1(\mu) = -D(\mu) + \sqrt{[R(\mu)]^2 - \|\hat{y}\|^2} \tag{10}$$

where

$$R(\mu) := \frac{\sin(\alpha + \varepsilon^{n-1}G(\mu))}{\cos(\alpha + \varepsilon^{n-1}G(\mu)) + \cos(\alpha + \tau/2)}$$

$$D(\mu) := \frac{\sin(\alpha + \tau/2)}{\cos(\alpha + \varepsilon^{n-1}G(\mu)) + \cos(\alpha + \tau/2)}.$$

Furthermore, the relation between μ and $\|\hat{y}\|$ is given by

$$\|\hat{y}\| = \frac{\sin(\alpha + \varepsilon^{n-1}G(\mu))\sin(\mu)}{1 + \cos(\alpha + \tau/2)\cos(\alpha + \varepsilon^{n-1}G(\mu)) + \sin(\alpha + \tau/2)\sin(\alpha + \varepsilon^{n-1}G(\mu))\cos(\mu)}. \tag{11}$$

By computing the derivative $\frac{d}{d\mu}\|\hat{y}\|$, one finds that the relation (11) is invertible in the region where both $\varepsilon^{n-1}G(\mu)$ and μ are small, and moreover that $\mu(\|\hat{y}\|) = 2\csc(\alpha)\cos^2(\tau/4)\|\hat{y}\| + \mathcal{O}(\|\hat{y}\|^3)$. Substituting this into (10) yields $y^1(\|\hat{y}\|) = G_\varepsilon(\|\hat{y}\|) := D(\mu(\|\hat{y}\|)) - \sqrt{[R(\mu(\|\hat{y}\|))]^2 - \|\hat{y}\|^2}$ from which one can find the asymptotic expansion

$$G_\varepsilon(\|\hat{y}\|)$$
$$= \begin{cases} -\tan(\tau/4) - \dfrac{\|\hat{y}\|^2}{2r} + \varepsilon c_2 - \varepsilon C_2 \log(\|\hat{y}\|) \\ \quad + \mathcal{O}(\|\hat{y}\|^4) + \mathcal{O}(\varepsilon\|\hat{y}\|^2 |\log(\|\hat{y}\|)|) & n = 2 \\[2mm] -\tan(\tau/4) - \dfrac{\|\hat{y}\|^2}{2r} + \dfrac{\varepsilon^2 C_3}{\|\hat{y}\|} + \mathcal{O}(\|\hat{y}\|^4) + \mathcal{O}\left(\varepsilon^2\|\hat{y}\|\right) & n = 3 \\[2mm] -\tan(\tau/4) - \dfrac{\|\hat{y}\|^2}{2r} + \dfrac{\varepsilon^3 C_4}{2\|\hat{y}\|^2} + \mathcal{O}(\|\hat{y}\|^4) + \mathcal{O}\left(\varepsilon^3[1 + |\log(\|\hat{y}\|)|]\right) & n = 4 \\[2mm] -\tan(\tau/4) - \dfrac{\|\hat{y}\|^2}{2r} + \dfrac{\varepsilon^{n-1} C_n}{(n-2)\|\hat{y}\|^{n-2}} + \mathcal{O}(\|\hat{y}\|^4) + \mathcal{O}\left(\dfrac{\varepsilon^{n-1}}{\|\hat{y}\|^{n-4}}\right) & n \geq 5 \end{cases} \tag{12}$$

in the region where both $\|\hat{y}\|$ and $\varepsilon^{n-1} G(\mu(\|\hat{y}\|))$ remain small. Here $d = d(\alpha, \tau)$ and $r = r(\alpha, \tau)$ are the quantities in (9), while

$$
\begin{aligned}
c_2 &= \frac{\cos(2\alpha + \tau/2)\big(1 + \log\big(\csc(\alpha)\cos^2(\tau/4)\big)\big)}{\big(\cos(\alpha) + \cos(\alpha + \tau/2)\big)^2} \\[2mm]
C_n &= \frac{\cos(2\alpha + \tau/2)}{\big(\cos(\alpha) + \cos(\alpha + \tau/2)\big)^2 \big(2\csc(\alpha)\cos^2(\tau/4)\big)^{n-2}}.
\end{aligned}
\tag{13}
$$

In a similar manner, one finds that the equation satisfied by the image under stereographic projection of the other hypersphere $R_{2\alpha+\tau}^{k+1}(\tilde{S}_\alpha)$ is $y^1(\|\hat{y}\|) = -G_\varepsilon(\|\hat{y}\|)$.

The task is now to find a truncation and re-scaling of the catenoid that 'fits' exactly within the gap between the two perturbed hyperspheres. The precise dimensions of this catenoid will be found by matching the asymptotic expansion of the graphing function given in (2) to the expansion (12) in the highest-order terms.

The upper and lower halves of the catenoid scaled by a factor $\tilde{\varepsilon}$ are the graphs $y^1 = \pm \tilde{\varepsilon} F(\|\hat{y}\|/\tilde{\varepsilon})$ where $F(x) = \int_1^x \big(\sigma^{2n-2} - 1\big)^{-1/2} d\sigma$. Hence one has the asymptotic expansion

$$
\tilde{\varepsilon} F(\|\hat{y}\|/\tilde{\varepsilon}) =
\begin{cases}
\tilde{\varepsilon} \log(2/\tilde{\varepsilon}) + \tilde{\varepsilon} \log(\|\hat{y}\|) - \dfrac{\tilde{\varepsilon}^3}{4\|\hat{y}\|^2} + \mathcal{O}\left(\dfrac{\tilde{\varepsilon}^5}{\|\hat{y}\|^4}\right) & n = 2 \\[4mm]
\tilde{\varepsilon} c_n - \dfrac{\tilde{\varepsilon}^{n-1}}{(n-2)\|\hat{y}\|^{n-2}} - \dfrac{\tilde{\varepsilon}^{3n-3}}{2(3n-4)\|\hat{y}\|^{3n-4}} + \mathcal{O}\left(\dfrac{\tilde{\varepsilon}^{5n-5}}{\|\hat{y}\|^{5n-6}}\right) & n \geq 3
\end{cases}
\tag{14}
$$

where c_n is yet another constant. If one compares the asymptotic expansion (14) (multiplied by -1) with the asymptotic expansion (12), then the matching is optimal if: $\tilde{\varepsilon} = \varepsilon$ in dimension $n = 2$ and $\tilde{\varepsilon} = \varepsilon C_n^{(n-2)/(n-1)}$ in higher dimensions; and ε is chosen to satisfy the equation $\tilde{\varepsilon} \log(2/\tilde{\varepsilon}) = \tan(\tau/4) + \varepsilon c_2$ in dimension $n = 2$ and $\varepsilon c = \tan(\tau/4)$ in higher dimensions.

The scale factor $\varepsilon := \varepsilon(\tau)$ is determined by the considerations above. Furthermore, the location where the catenoid must be truncated in order for the gluing to take place with optimal error can be determined by similar considerations. That is, once ε and $\tilde{\varepsilon}$ have been found, then the error $|F(\|\hat{y}\|) - G_\varepsilon(\|\hat{y}\|)|$ is the smallest when $\hat{y}$ is chosen to lie in a range where the quantity $\frac{1}{2r}\|\hat{y}\|^2 + \frac{\varepsilon^{3n-3}}{2(3n-4)}\|\hat{y}\|^{4-3n}$ is minimized. This occurs when $\|\hat{y}\| = \mathcal{O}(\rho_\varepsilon)$ where $\rho_\varepsilon := \varepsilon^{(3n-3)/(3n-2)}$. Thus one must truncate the ε-scaled catenoid exactly at $\|\hat{y}\| = \rho_\varepsilon$ for an optimally smooth gluing.

2.3.5 Assembling the approximate solution

The first step is to construct a smooth hypersurface with boundary in $\mathbb{R}^{n+1}$ that interpolates between the stereographic coordinate images of a pair of perturbed hyperspheres near the y^1-axis. Denote these objects by $S_\pm$. Now let $\eta : [0, \infty) \to \mathbb{R}$ be a smooth, monotone cut-off function satisfying $\eta(s) = 0$ for $s \in [0, 1/2]$ and $\eta(s) = 1$ for $s \in [2, \infty)$. Define the function $\tilde{F}_{\alpha, \tau} : \bar{B}_{2\rho_\varepsilon}(0) \setminus B_\varepsilon(0) \subseteq \mathbb{R}^n \to \mathbb{R}$ by

$$
\tilde{F}_{\alpha, \tau}(\hat{y}) = \varepsilon\big(1 - \eta(\|\hat{y}\|/\rho_\varepsilon)\big) F(\|\hat{y}\|/\varepsilon) + \eta(\|\hat{y}\|/\rho_\varepsilon) G_\varepsilon(\|\hat{y}\|).
\tag{15}
$$

Then define the hypersurfaces $\tilde{\Sigma}_\varepsilon^\pm = \{(\pm \tilde{F}_{a, \tau}(\hat{y}), \hat{y}) : \|\hat{y}\| \in [\varepsilon, 2\rho_\varepsilon]\}$ so that $\tilde{\Sigma}_\varepsilon := \tilde{\Sigma}_\varepsilon^+ \cup \tilde{\Sigma}_\varepsilon^-$ is a smooth hypersurface connecting $S_+ \setminus (\mathbb{R} \times B_{2\rho_\varepsilon}(0))$ to $S_- \setminus (\mathbb{R} \times B_{2\rho_\varepsilon}(0))$ through the catenoid.

Pushing forward to the sphere, one now has a small catenoidal hypersurface $R_{\alpha+\tau/2}^{2k+1} \circ K^{-1}(\tilde{\Sigma}_\varepsilon)$ that overlaps perfectly with the pair of perturbed hyperspheres $R_{2\alpha+\tau}^k(\tilde{S}_\alpha)$ and $R_{2\alpha+\tau}^{k+1}(\tilde{S}_\alpha)$ inside a small tubular annulus of the geodesic γ near the point $R_{\alpha+\tau/2}^{2k+1}(p)$.

The second step is to connect the hypersurfaces above together according to the following definition. Note that there exists a radius $\tilde{\rho}_\varepsilon$ so that the boundary of $R_{\alpha+2\tau}^k(S_\alpha \backslash B_{\tilde{\rho}_\varepsilon}(p^+)) \cup R_{\alpha+2\tau}^{k+1}(S_\alpha \backslash B_{\tilde{\rho}_\varepsilon}(p^-))$ under the stereographic projection $R_{\alpha+\tau/2}^{2k+1} \circ K$ coincides with $\partial(\tilde{\Sigma}_\varepsilon^+ \cup \tilde{\Sigma}_\varepsilon^-)$. Clearly $\tilde{\rho}_\varepsilon = \mathcal{O}(\rho_\varepsilon)$.

Definition 5 The *approximate solution* with parameters α and τ is the hypersurface

$$\tilde{\Lambda}_{\alpha,\tau} := \left[\bigcup_{k=0}^\infty R_{2\alpha+\tau}^k\left(\tilde{S}_\alpha \backslash (B_{\tilde{\rho}_\varepsilon}(p^+) \cup B_{\tilde{\rho}_\varepsilon}(p^-))\right)\right] \cup \left[\bigcup_{k=0}^\infty R_{\alpha+\tau/2}^{2k+1} \circ K^{-1}(\tilde{\Sigma}_\varepsilon)\right] \quad (16)$$

where $\varepsilon = \varepsilon(\tau)$ is the scale parameter associated to τ.

Note that when the angle $2\alpha + \tau$ is a rational multiple of 2π, then $\tilde{\Lambda}_{\alpha,\tau}$ is a compact immersed hypersurface of $\mathbb{S}^n$ that winds around the geodesic γ an integral number of times. In this case, $R_{2\alpha+\tau}$ generates a finite cyclic subgroup of $SO(n+1)$ of order N and the infinite unions in (16) can be replaced by finite unions from $k = 0$ to $k = N - 1$. Furthermore, $\tilde{\Lambda}_{\alpha,\tau}$ is embedded if and only if it wraps around γ exactly once. If $2\alpha + \tau$ is not a rational multiple of 2π, then $\tilde{\Lambda}_{\alpha,\tau}$ is non-compact. It will henceforth be assumed that $2\alpha + \tau$ is a rational multiple of 2π. The analysis that follows can readily be adapted to the case of general $2\alpha + \tau$.

This section of this article concludes with the following identification of the various regions of $\tilde{\Lambda}_{\alpha,\tau}$. Let $Cyl(\rho) := \{(y^1, \hat{y}) \in \mathbb{R} \times \mathbb{R}^n : \|\hat{y}\| < \rho\}$ denote the cylinder of radius ρ parallel to the y^1-axis in $\mathbb{R} \times \mathbb{R}^n$.

Definition 6 The approximate solution $\tilde{\Lambda}_{\alpha,\tau}$ is divided into the following three regions.

- Let $\mathcal{N}_\varepsilon^k := R_{\alpha+\tau/2}^{2k+1} \circ K^{-1}\left(\tilde{\Sigma}_\varepsilon \cap Cyl(\rho_\varepsilon/2)\right)$. Then $\mathcal{N}_\varepsilon := \bigcup_{k=0}^\infty \mathcal{N}_\varepsilon^k$ is the *neck region* of $\tilde{\Lambda}_{\alpha,\tau}$.
- Let $\mathcal{T}_\varepsilon^{k,\pm} := R_{\alpha+\tau/2}^{2k+1} \circ K^{-1}\left(\tilde{\Sigma}_\varepsilon^\pm \cap \left[Cyl(2\rho_\varepsilon) \backslash Cyl(\rho_\varepsilon/2)\right]\right)$. Then $\mathcal{T}_\varepsilon := \bigcup_{k=0}^\infty \left[\mathcal{T}_\varepsilon^{k,+} \cup \mathcal{T}_\varepsilon^{k,-}\right]$ is the *transition region* of $\tilde{\Lambda}_{\alpha,\tau}$.
- Let $\mathcal{E}_\varepsilon^k := R_{2\alpha+\tau}^k\left(\tilde{S}_\alpha \backslash (B_{\tilde{\rho}_\varepsilon}(p^+) \cup B_{\tilde{\rho}_\varepsilon}(p^-))\right)$. Then $\mathcal{E}_\varepsilon := \bigcup_{k=0}^\infty \mathcal{E}_\varepsilon^k = \tilde{\Lambda}_{\alpha,\tau} \backslash \left[\mathcal{N}_\varepsilon \cup \mathcal{T}_\varepsilon\right]$ is the *exterior region* of $\tilde{\Lambda}_{\alpha,\tau}$.

2.4 Symmetries of the approximate solution

It is clear that $\tilde{\Lambda}_{\alpha,\tau}$ is invariant under the group generated by $R_{2\alpha+\tau}$. The additional symmetries possessed by $\tilde{\Lambda}_{\alpha,\tau}$ will play an important role in the forthcoming analysis. The first set of symmetries is defined as follows. Define $S_B^{01} \in O(n+2)$ by choosing $B \in O(n)$ and then setting

$$S_B^{01} = \left(\begin{array}{c|c} \begin{matrix} 1 & \\ & 1 \end{matrix} & 0 \\ \hline 0 & B \end{array}\right)$$

which is the transformation keeping the x^0 and x^1 coordinates fixed while rotating the remaining coordinates by B. That each S_B^{01} is a symmetry of $\tilde{\Lambda}_{\alpha,\tau}$ can be seen as follows. First, S_B^{01} commutes with $R_{2\alpha+\tau}$ so that it is sufficient to check that one perturbed hypersphere, one neck region $\mathcal{N}_\varepsilon^0$ and one pair of transition regions $\mathcal{T}_\varepsilon^{0,\pm}$ are preserved by S_B^{01}. But this

holds because the construction of the interpolating catenoidal neck guarantees that rotational symmetry around the y^1-axis in the stereographic coordinates is preserved.

One additional symmetry must be identified. Define the reflection $T \in O(n+2)$ by

$$T(x^0, x^1, x^2, \ldots, x^{n+1}) = (x^0, -x^1, x^2, \ldots, x^{n+1}).$$

That T is also a symmetry of $\tilde{\Lambda}_{\alpha,\tau}$ can be seen as follows. First, $T \circ R_{2\alpha+\tau} = R_{2\alpha+\tau}^{-1} \circ T$ so that T permutes the perturbed hyperspheres and the interpolating catenoidal necks amongst each other, perhaps with opposite orientation. The change of orientation does not matter for the perturbed hyperspheres because the graphing function G satisfies $G(\mu) = G(\pi - \mu)$. Moreover, the catenoidal necks and transition regions were constructed so as to be preserved under the transformation $y^1 \longmapsto -y^1$ in the stereographic coordinates. Hence the change of orientation does not matter for these either. Thus T is also a symmetry of $\tilde{\Lambda}_{\alpha,\tau}$.

2.5 The analytic set-up

2.5.1 Deforming the approximate solution

The approximate solution $\tilde{\Lambda}_{\alpha,\tau}$ constructed in the Sect. 2.3 has mean curvature almost equal to H_α everywhere except in the neck and transition regions where the mean curvature transitions to zero (but perhaps in non-uniform way due to the errors introduced by the cut-off functions). The next task is to develop a means for deforming $\tilde{\Lambda}_{\alpha,\tau}$ into an exactly CMC hypersurface, with mean curvature equal to H_α.

Since $\tilde{\Lambda}_{\alpha,\tau}$ is a hypersurface, it is possible to parametrize deformations of $\tilde{\Lambda}_{\alpha,\tau}$ in a very standard way via normal deformations. These can be constructed by choosing a function $f : \tilde{\Lambda}_{\alpha,\tau} \to \mathbb{R}$ and then considering the deformation $\phi_f : \tilde{\Lambda}_{\alpha,\tau} \to \mathbb{S}^{n+1}$ given by $\phi_f(q) := \exp_q(f(q) \cdot N(q))$ where $\exp_q$ is the exponential map at the point q and $N(q)$ is the outward unit normal vector field of $\tilde{\Lambda}_{\alpha,\tau}$ at the point q. For any given function f, the hypersurface $\phi_f(\tilde{\Lambda}_{\alpha,\tau})$ is a normal graph over $\tilde{\Lambda}_{\alpha,\tau}$, provided f is sufficiently small in a C^1 sense. Finding an exactly CMC normal graph near $\tilde{\Lambda}_{\alpha,\tau}$ therefore consists of finding a function f satisfying the equation $H_{\phi_f(\tilde{\Lambda}_{\alpha,\tau})} = H_\alpha$, where H_Λ denotes the mean curvature of a hypersurface Λ.

Definition 7 Let $\Phi_{\alpha,\tau}$ be the operator $f \longmapsto H_{\phi_f(\tilde{\Lambda}_{\alpha,\tau})} - H_\alpha$.

This is a quasi-linear, second-order partial differential operator for the function f whose zero gives the desired deformation of $\tilde{\Lambda}_{\alpha,\tau}$.

2.5.2 The Banach space inverse function theorem

Finding a solution of the equation $\Phi_{\alpha,\tau}(f) = 0$ when τ is sufficiently small will be accomplished by invoking the *Banach space inverse function theorem*. To provide a focus for the remainder of the proof of Main Theorem 1, this fundamental result will be stated here in fairly general terms. See [1] for a proof.

Theorem (IFT) *Let* $\Phi : X \to Z$ *be a smooth map of Banach spaces, set* $\Phi(0) := E$ *and define the linearized operator* $\mathcal{L} := \mathrm{D}\Phi(f) = \frac{\mathrm{d}}{\mathrm{d}s}\Phi(f + su)\big|_{s=0}$. *Suppose* $\mathcal{L}$ *is bounded and either* $\mathcal{L}$ *is invertible and satisfies*

$$\|\mathcal{L}^{-1}(z)\| \le C \|z\| \tag{17a}$$

for all $z \in Z$; or else $\mathcal{L}$ is surjective and possesses a bounded right inverse $\mathcal{R} : Z \to X$ satisfying

$$\|\mathcal{R}(z)\| \le C\|z\| \tag{17b}$$

for all $z \in Z$. Choose ρ so that if $y \in B_\rho(0) \subseteq X$, then

$$\|\mathcal{L}(x) - D\Phi(y)(x)\| \le \frac{1}{2C}\|x\| \tag{18}$$

for all $x \in X$, where $C > 0$ is a constant. Then if $z \in Z$ is such that

$$\|z - E\| \le \frac{\rho}{2C}, \tag{19}$$

there exists a unique $x \in B_\rho(0)$ so that $\Phi(x) = z$. Moreover, $\|x\| \le 2C\|z - E\|$.

As the statement of theorem makes clear, it must be the case that $\tilde{\mathcal{L}}_{\alpha,\tau}$ is surjective in order to solve the equation $\Phi_{\alpha,\tau}(f) = 0$ in a Banach subspace X of at least C^2 functions on $\tilde{\Lambda}_{\alpha,\tau}$. In addition, it is necessary to establish the three fundamental estimates (17), (18) and (19) of the theorem. It will turn out that $D\Phi_{\alpha,\tau}(0)$ is invertible in the setting of Main Theorem 1 and the first of these estimates, called the *linear estimate* is proved in Sect. 2.7 after some preliminary work in Sects. 2.6 and 2.5 that helps identify the correct Banach subspace X. The second estimate determines the variation of the operator $D\Phi_{\alpha,\tau}(f)$ as f varies. The third estimate is of the size of $\Phi_{\alpha,\tau}(0)$ in the norm of Z. The latter two estimates, called the *non-linear estimates*, are proved in Sect. 2.8. In all the three estimates, it will be necessary to determine explicitly the τ-dependence of the constant C in order to apply them in a uniform way when τ is close to zero.

2.5.3 The linearized operator and the Jacobi fields

In order to solve the equation $\Phi_{\alpha,\tau}(f) = 0$, the linearized operator $D\Phi_{\alpha,\tau}(0)$ (hereinafter, abbreviated $\tilde{\mathcal{L}}_{\alpha,\tau}$) must satisfy two fundamental requirements: it must be surjective; and it must possess a right inverse bounded above by a constant independent of τ. It is, however, a general phenomenon in singular perturbation problems that the linearized operator often has a kernel and a co-kernel, as well as small eigenvalues tending to zero as the singular parameter (in this case τ) tends to zero. These are the *obstructions* preventing the deformation to an exactly CMC hypersurface.

The origin of these obstructions in the present case is geometric. To see this, recall the general fact that any one-parameter family of isometries of the ambient space in which a CMC hypersurface is situated gives rise to an element in the kernel of the linearized mean curvature operator as follows.

Lemma 8 *Let Λ be a closed hypersurface in a Riemannian manifold X with mean curvature H_Λ, second fundamental form B_Λ and unit normal vector field N_Λ. If R_t is a one-parameter family of isometries of X with deformation vector field $V = \frac{d}{dt}R_t\big|_{t=0}$, then the function $q_V := \langle V, N_\Lambda \rangle$ belongs to the kernel of Λ.*

Proof Since R_t is a family of isometries, then $H(R_t(\Lambda)) = H(\Lambda)$ for all t and $\frac{d}{dt}\big|_{t=0} H(R_t(\Lambda)) = 0$. The function $q_V := \langle V, N_\Lambda \rangle$ is thus in the kernel of $DH_\Lambda(0)$ because q_V generates a normal deformation of Λ whose action, to first order, coincides with R_t. $\qquad\square$

Both kinds of obstructions to the solvability of the equation $\Phi_{\alpha,\tau}(f) = 0$ on $\tilde{\Lambda}_{\alpha,\tau}$ (namely those corresponding to elements in the kernel and those corresponding to small eigenvalues) can be explained using Lemma 8. First of all, since rotations of $\mathbb{S}^{n+1}$ are isometries, every one-parameter rotation of $\tilde{\Lambda}_{\alpha,\tau}$ generates a function in the kernel of $\tilde{\mathcal{L}}_{\alpha,\tau}$ exactly as in the lemma. These functions are known as *Jacobi fields*. Second of all, one can imagine transformations of $\tilde{\Lambda}_{\alpha,\tau}$ which rotate exactly one of its constituent hyperspheres or catenoidal necks by a rotation in $SO(n+2)$ while leaving all the other constituent hyperspheres and necks fixed. The associated function q_V coincides with an exact Jacobi field on the constituent being transformed and vanishes on the other constituents of $\tilde{\Lambda}_{\alpha,\tau}$. It is well known that the linear span of these functions approximates the small eigenspaces of $\tilde{\mathcal{L}}_{\alpha,\tau}$ [12, Appendix B]. These functions will be called *approximate Jacobi fields*.

The existence of these two types of obstructions makes it impossible to meet the requirements for solving for the perturbation of $\tilde{\Lambda}_{\alpha,\tau}$ into a CMC hypersurface. One way to avoid these obstructions is to exploit the natural symmetry of $\tilde{\Lambda}_{\alpha,\tau}$ and then to deform $\tilde{\Lambda}_{\alpha,\tau}$ equivariantly; i.e., by normal deformations corresponding to functions on $\tilde{\Lambda}_{\alpha,\tau}$ that are invariant under these symmetries. The key is to have enough symmetries so that there are no Jacobi fields—either approximate or true—that are invariant with respect to all of these symmetries at once. That this situation holds for $\tilde{\Lambda}_{\alpha,\tau}$ will be shown in Sect. 2.7.

To continue, it will be necessary to have an explicit representation of the Jacobi fields and approximate Jacobi fields of $\tilde{\Lambda}_{\alpha,\tau}$. First, it is clear what the true Jacobi fields of $\tilde{\Lambda}_{\alpha,\tau}$ look like: they are of the form $\langle V, N \rangle$ where V generates a rotation and N is the unit normal vector field of $\tilde{\Lambda}_{\alpha,\tau}$. Second, according to [12, Appendix B], the approximate Jacobi fields can be constructed by multiplying the function $\langle V, N \rangle$ by a cut-off function whose support is exactly one of the constituent hyperspheres or catenoidal necks of $\tilde{\Lambda}_{\alpha,\tau}$. It is thus necessary to understand the functions $\langle V, N \rangle$, as V ranges over all generators of $SO(n+2)$, restricted to a hypersphere or to a catenoidal neck. What follows is a presentation of these functions.

1. *Jacobi fields of the hyperspheres.*
   The linearized mean curvature operator of S_α is easily computed to be

   $$\mathcal{L}_a := \sin^{-2}(\alpha)\big(\Delta_{\mathbb{S}^n} + n\big).$$

   Therefore, the Jacobi fields of $\mathcal{L}_a$ are simply the eigenfunctions of the n-sphere with eigenvalue n. In the context of Lemma 8, these can be derived by considering all the non-trivial rotations of S_α, namely, the rotations generated by the vector fields

   $$V_k := x^k \frac{\partial}{\partial x^0} - x^0 \frac{\partial}{\partial x^k} \quad \text{for } k = 1, \ldots, n+1.$$

   Taking the inner product of V_k with N_α and restricting the resulting function to S_α gives a Jacobi field for each k. This procedure yields the coordinate functions x^k restricted to S_α.

2. *Jacobi fields of the catenoidal necks.*
   The catenoidal necks of $\tilde{\Lambda}_{\alpha,\tau}$ are catenoids Σ in $\mathbb{R}^{n+1}$ that have been re-scaled and embedded in $\mathbb{S}^n$ by the inverse of the canonical stereographic projection. It turns out that when the scale parameter ε is sufficiently small, it is sufficient to consider the Jacobi fields of Σ treated as a hypersurface in $\mathbb{R}^{n+1}$, where the ambient metric is Euclidean (rather than the one induced by stereographic projection from the sphere) and the ambient isometries are the rigid motions of $\mathbb{R}^{n+1}$. The linearized mean curvature operator of Σ with respect to this background is easily computed to be

$$\mathcal{L}_\Sigma := \frac{1}{\phi^n} \frac{\partial}{\partial s}\left(\phi^{n-2}\frac{\partial}{\partial s}\right) + \frac{1}{\phi^2}\Delta_{\mathbb{S}^{n-1}} + \frac{n(n-1)}{\phi^{2n}}$$

in the standard parametrization of Definition 3. The isometries generating the relevant Jacobi fields of Σ are as follows. First, the ambient space $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ possesses n translations along the $\mathbb{R}^n$ factor and one translation in the $\mathbb{R}$ direction, which are generated by the vector fields

$$V_k^{\text{trans}} := \frac{\partial}{\partial y^k} \quad \text{for } k = 1, \ldots, n+1.$$

Then there are n rotations of $\mathbb{R} \times \mathbb{R}^n$ that do not preserve the $\mathbb{R}$-direction, which are generated by the vector fields

$$V_{1k}^{\text{rot}} := y^1 \frac{\partial}{\partial y^k} - y^k \frac{\partial}{\partial y^1} \quad \text{for } k = 2, \ldots, n+1.$$

Finally, the motion of dilation in $\mathbb{R}^{n+1}$, though not an isometry, does preserve the mean curvature zero condition and is thus a geometric motion to which Lemma 8 can be applied. Dilation is generated by the vector field

$$V^{\text{dil}} := \sum_{k=1}^{n+1} y^k \frac{\partial}{\partial y^k}.$$

The Jacobi fields of $\mathcal{L}_\Sigma$ arising from the three classes of motions above can be found by the procedure of Lemma 8. One obtains the following non-trivial functions:

$$
\begin{aligned}
J_1(s) \quad &:= \langle N_\Sigma, V_1^{\text{trans}} \rangle = \frac{\dot{\phi}(s)}{\phi(s)} \\
J_k(s, \Theta) \quad &:= \langle N_\Sigma, V_k^{\text{trans}} \rangle = -\frac{\Theta^k}{\phi^{n-1}(s)} \quad k = 2, \ldots, n+1 \\
J_{1k}(s, \Theta) \quad &:= \langle N_\Sigma, V_{1k}^{\text{rot}} \rangle = \Theta^k\left(\frac{\psi(s)}{\phi^{n-1}(s)} + \dot{\phi}(s)\right) \quad k = 2, \ldots, n+1 \\
J_0(s) \quad &:= \langle N_\Sigma, V^{\text{dil}} \rangle = \frac{\psi(s)\dot{\phi}(s)}{\phi(s)} - \frac{1}{\phi^{n-2}(s)}.
\end{aligned}
\tag{20}
$$

Note that the functions J_k with $k \neq 0$ have odd symmetry with respect to the central sphere of Σ, i.e., with respect to the transformation $s \mapsto -s$, whereas J_{1k} and J_0 have even symmetry. Note also that J_1 is bounded while J_0 has linear growth when the dimension is $n = 2$ and is bounded in higher dimensions; J_k decays like $\exp(-(n-1)|s|)$ for large $|s|$; and J_{1k} grows like $\exp(|s|)$ for large $|s|$.

2.6 Function spaces and norms

It does not seem possible to obtain a 'good' linear estimate of the form $\|\tilde{\mathcal{L}}_{\alpha,\tau}(u)\| \geq C\|u\|$ with any straightforward choice of Banach subspaces and norms, where 'good' in this case means it has a constant C independent of τ, due to the presence of Jacobi fields and approximate Jacobi fields. In order to remedy this situation, one must implement two ideas. As mentioned above, the first of these ideas is to impose symmetry conditions on the functions chosen for deforming $\tilde{\Lambda}_{\alpha,\tau}$ which have the effect of ruling out the Jacobi fields. The second

idea is to introduce a special norm to measure the 'size' of functions $f \in C^{2,\beta}(\tilde{\Lambda}_{\alpha,\tau})$ in order to properly determine the dependence on the parameter τ of the various estimates needed for the application of the inverse function theorem.

The norm in question is the so-called *weighted Schauder norm*. To define this norm, one must first define a *weight function* on $\tilde{\Lambda}_{\alpha,\tau}$. Let $\varepsilon = \varepsilon(\tau)$ be the scale parameter of $\tilde{\Lambda}_{\alpha,\tau}$, recall the definitions from Sect. 2.3 and fix some ρ_0 independent of τ satisfying $\rho_o \gg 2\rho_\varepsilon$ such that the balls of radii $2\rho_0$ centred on two different neck regions do not intersect.

Definition 9 The *weight function* $\zeta_\varepsilon : \tilde{\Lambda}_{\alpha,\tau} \to \mathbb{R}$ is defined by

$$
\zeta_\varepsilon(q) = \begin{cases}
\varepsilon \cosh(s) & q = R_{\alpha+\tau/2}^{2k+1} \circ K^{-1}(\varepsilon\psi(s), \varepsilon\phi(s)\Theta) \in \mathcal{N}_\varepsilon^k \\
Interpolation & q \in \mathcal{T}_\varepsilon \\
\mathrm{dist}(q, \gamma) & q \in \mathcal{E}_\varepsilon \cap \left[\bigcup_{k=0}^{N-1} B_{\rho_0}\left(R_{\alpha+\tau/2}^{2k+1}(p) \right) \right] \\
Interpolation & q \in \mathcal{E}_\varepsilon \cap \left[\bigcup_{k=0}^{N-1} \left(B_{2\rho_0}\left(R_{\alpha+\tau/2}^{2k+1}(p) \right) \backslash B_{\rho_0}\left(R_{\alpha+\tau/2}^{2k+1}(p) \right) \right) \right] \\
\rho_0 & q \in \mathcal{E}_\varepsilon \backslash \left[\bigcup_{k=0}^{N-1} B_{2\rho_0}\left(R_{\alpha+\tau/2}^{2k+1}(p) \right) \right].
\end{cases}
$$

The interpolation is such that ζ_ε is smooth and monotone, and is such that ζ_ε is invariant under the symmetries of $\tilde{\Lambda}_{\alpha,\tau}$. The weighted Schauder norm is then defined as follows. (The weighted norm below is essentially an adaptation of the weighted norm used in [24].) Let T be any tensor on $\tilde{\Lambda}_{\alpha,\tau}$ and let $\mathcal{U} \subseteq \tilde{\Lambda}_{\alpha,\tau}$ be any open subset. Recall the notation

$$
|T|_{0,\mathcal{U}} = \sup_{q \in \mathcal{U}} \|T(q)\|
$$

$$
\text{and} \quad [T]_{\beta,\mathcal{U}} = \sup_{q,q' \in \mathcal{U}} \frac{\|T(q') - \Xi_{q,q'}(T(q))\|}{\mathrm{dist}(q, q')^\beta},
$$

where the norms and the distance function that appear are taken with respect to the induced metric of $\tilde{\Lambda}_{\alpha,\tau}$, while $\Xi_{q,q'}$ is the corresponding parallel transport operator from q to q'. Now let $Tub_\rho(\gamma)$ be the tubular neighbourhood of γ having width ρ; and for any $\arctan(\varepsilon/2) < \rho < \rho_0$ define the annular region $A_\rho = \tilde{\Lambda}_{\alpha,\tau} \cap \left[Tub_\rho(\gamma) \backslash Tub_{\rho/2}(\gamma) \right]$. Then the norm on any subset $\mathcal{U} \subseteq A_\rho$ is

$$
|f|_{l,\beta,\delta,\mathcal{U} \cap A_\rho} := \zeta_\varepsilon^{-\delta}(\rho)|f|_{0,\mathcal{U} \cap A_\rho} + \cdots + \zeta_\varepsilon^{-\delta+l}(\rho)|\nabla^l f|_{0,\mathcal{U} \cap A_\rho}
$$
$$
+ \zeta_\varepsilon^{-\delta+l+\beta}(\rho)[\nabla^l f]_{\beta,\mathcal{U} \cap A_\rho}.
$$

Now make the following definition.

Definition 10 Let $\mathcal{U} \subset \tilde{\Lambda}_{\alpha,\tau}$. The $C_\delta^{l,\beta}$ norm of a function defined on $\mathcal{U}$ is given by

$$
|f|_{C_\delta^{l,\beta}(\mathcal{U})} := \sum_{i=0}^{l} |\nabla^i f|_{0,\mathcal{U} \cap \left[\tilde{\Lambda}_{\alpha,\tau} \backslash N_{\rho_0}(\gamma) \right]} + [\nabla^l f]_{\beta,\mathcal{U} \cap \left[\tilde{\Lambda}_{\alpha,\tau} \backslash N_{\rho_0} \right]}
$$
$$
+ \sup_{\rho \in (2\varepsilon, \rho_0]} |f|_{l,\beta,\delta,\mathcal{U} \cap A_\rho}. \tag{21}
$$

The notation for the norm $|\cdot|_{C_\delta^{l,\beta}(\tilde{\Lambda}_{\alpha,\tau})}$ will be abbreviated $|\cdot|_{C_\delta^{l,\beta}}$, but $C_\delta^{l,\beta}$ norms over other subsets will be written out in full. Now let X be any space of tensor fields. The Banach space $C_\delta^{k,\beta}(X)$ denotes the $C^{l,\beta}$ tensor fields in X measured with respect to the norm (21). Finally, it is well known that all the 'usual' properties that one would expect from a Schauder norm

(multiplicative properties, interpolation inequalities and so on) are satisfied by the weighted $C_\delta^{k,\beta}$ norms.

The choice of the Banach space in which a solution of the deformation problem will be found is governed by the following considerations. The functions must be $C^{2,\beta}$ and they must be measured with respect to the $C_\delta^{2,\beta}$ norm. Also, they must be invariant with respect to all the symmetries of $\tilde{\Lambda}_{\alpha,\tau}$ since this will have the effect of ruling out the presence of non-invariant Jacobi fields. The following space meets all these needs.

Definition 11 Let $X := \{f : \tilde{\Lambda}_{\alpha,\tau} \to \mathbb{R} : f \circ R_{2\alpha+\tau} = f \circ T = f \text{ and } f \circ S_B^{01} = f \quad \forall B \in SO(n)\}$.

Since $R_{2\alpha+\tau}$, T and all S_B^{01} are isometries of $\mathbb{S}^n$, then $\Phi_{\alpha,\tau}(f) \circ R_{2\alpha+\tau} = \Phi_{\alpha,\tau}(f \circ R_{2\alpha+\tau}) = \Phi_{\alpha,\tau}(f)$ and similarly $\Phi_{\alpha,\tau}(f) \circ S_B^{01} = \Phi_{\alpha,\tau}(f) \circ T = \Phi_{\alpha,\tau}(f)$ whenever $f \in X$. Hence $\Phi_{\alpha,\tau}$ can by symmetrized to yield a new operator (which will be given the same name) $\Phi_{\alpha,\tau} : X \to X$. In addition, it is easy to verify that $\tilde{\mathcal{L}}_{\alpha,\tau} : C_\delta^{2,\beta}(X) \to C_{\delta-2}^{0,\beta}(X)$ is a bounded operator whose operator norm is bounded above by a constant independent of τ.

2.7 The linear estimate

The most involved estimate necessary for invoking the inverse function theorem in the proof of Main Theorem 1 is the *linear estimate*, i.e., the estimate (17) of the linearization $\tilde{\mathcal{L}}_{\alpha,\tau}$ from below. The purpose of this section is to prove this estimate in the Banach space $C_\delta^{2,\beta}(X)$. Because the second-order part of $\tilde{\mathcal{L}}_{\alpha,\tau}$, namely, the Laplace operator, has a double indicial root when the dimension of $\tilde{\Lambda}_{\alpha,\tau}$ is $n = 2$, the proof of the estimate is more complicated in this dimension. Thus the easier case of dimension $n \geq 3$ will be given first. The method will be to construct an explicit solution of the equation $\tilde{\mathcal{L}}_{\alpha,\tau}(u) = f$ by patching together local solutions on the neck region and away from the neck region.

Proposition 12 *Suppose $n \geq 3$ and choose $\delta \in (2-n, 0)$ and $\tau > 0$ sufficiently small. The linearized operator $\tilde{\mathcal{L}}_{\alpha,\tau} : C_\delta^{2,\beta}(X) \to C_{\delta-2}^{0,\beta}(X)$ satisfies the estimate*

$$|\tilde{\mathcal{L}}_{\alpha,\tau}(u)|_{C_{\delta-2}^{0,\beta}} \geq C|u|_{C_\delta^{2,\beta}}$$

where C is a constant independent of τ.

Proof The patching argument requires two carefully defined partitions of unity for the various pieces of $\tilde{\Lambda}_{\alpha,\tau}$. First, for any $\rho \in (\tilde{\rho}_\varepsilon, \rho_0)$, define $\mathcal{E}_\varepsilon^k(\rho) := \mathcal{E}_\varepsilon^k \setminus Tub_\rho(\gamma)$, and define $\mathcal{N}_\varepsilon^k(\rho)$ to be the disjoint component of $Tub_\rho(\gamma)$ containing $\mathcal{N}_\varepsilon^k$. Next, define the smooth, monotone cut-off functions

$$\chi_{ext,\rho}^k(q) := \begin{cases} 1 & q \in \mathcal{E}_\varepsilon^k(2\rho) \\ 0 & q \in \tilde{\Lambda}_{\alpha,\tau} \cap Tub_\rho(\gamma) \end{cases}$$

$$\chi_{neck,\rho}^k(q) := \begin{cases} 1 & q \in \mathcal{N}_\varepsilon^k(2\rho) \\ 0 & q \in \tilde{\Lambda}_{\alpha,\tau} \setminus Tub_\rho(\gamma) \end{cases}$$

so that $\sum_{k=0}^{N-1} \chi_{ext,\rho}^k + \sum_{k=0}^{N-1} \chi_{neck,\rho}^k = 1$. Finally, define another set of cut-off functions

$$\eta_{ext}^k(q) := \begin{cases} 1 & q \in \mathcal{E}_\varepsilon^k \\ 0 & q \in \tilde{\Lambda}_{\alpha,\tau} \setminus [\mathcal{T}_\varepsilon^{k-1,+} \cup \mathcal{E}_\varepsilon^k \cup \mathcal{T}_\varepsilon^{k,-}] \end{cases}$$

$$\eta^k_{neck}(q) \quad := \quad \begin{cases} 1 & q \in \mathcal{N}^k_\varepsilon \\ 0 & q \in \tilde{\Lambda}_{\alpha,\tau} \backslash [\mathcal{T}^{k,-}_\varepsilon \cup \mathcal{N}^k_\varepsilon \cup \mathcal{T}^{k,+}_\varepsilon] \end{cases}$$

so that once again $\sum_{k=0}^{N-1} \eta^k_{ext} + \sum_{k=0}^{N-1} \eta^k_{neck} = 1$. In addition, one can assume that these cut-off functions are invariant under all the desired symmetries.

Suppose now that $f \in C^{0,\beta}_{\delta-2}(X)$ is given. The solution of the equation $\tilde{\mathcal{L}}_{\alpha,\tau}(u) = f$ will be constructed in three stages: local solutions on the neck regions will be found; then local solutions on the exterior regions will be found; and finally these solutions will be patched together to form an approximate solution which can be perturbed to a solution by iteration. To begin this process, fix a small $\rho \in (\tilde{\rho}_\varepsilon, \rho_0)$ and write $f = \sum_{k=0}^{N-1} f^k_{ext} + \sum_{k=0}^{N-1} f^k_{neck}$ where $f^k_{ext} := f \cdot \chi^k_{ext,\rho}$ and $f^k_{neck} := f \cdot \chi^k_{neck,\rho}$. The various symmetries satisfied by f ensure that $f^k_{ext,\rho} := f_{ext,\rho}$ and $f^k_{neck,\rho} := f_{neck,\rho}$ for all k. Also, $f_{*,\rho} \circ S^{01}_B = f_{*,\rho}$ for all $B \in O(n)$ and $f_{*,\rho} \circ T = f_{*,\rho}$.

Step 1. Local solutions on the neck regions. For each k, the subset $K \circ R^{-(2k+1)}_{\alpha+\tau/2}(\mathcal{N}^k_\varepsilon(\rho)) \subseteq \mathbb{R} \times \mathbb{R}^n$ is the union of two graphs over the annulus $\{\hat{y} : \varepsilon \leq \|\hat{y}\| \leq \rho'\} \subseteq \mathbb{R}^n$ for some ρ' which is small but independent of ε. The graphing functions are $y^1 = \pm \tilde{F}_\varepsilon(\|\hat{y}\|)$ as defined in Sect. 2.3. Furthermore, $K \circ R^{-(2k+1)}_{\alpha+\tau/2}(\mathcal{N}^k_\varepsilon(\rho))$ is a perturbation of the ε-scaled catenoid $\varepsilon\Sigma$. Consequently, the function f_{neck} and the equation $\tilde{\mathcal{L}}_{\alpha,\tau}(u) = f_{neck}$ can be pulled back to a compact subset of $\varepsilon\Sigma$. In this formulation, this compact subset carries a perturbation of the catenoid metric $4\varepsilon^2 g_\Sigma$. However, f_{neck} will be viewed as a function of compact support on the complete catenoid $\varepsilon\Sigma$ carrying exactly the metric $4\varepsilon^2 g_\Sigma$, and the equation that will be solved in this step is $\frac{1}{4}\mathcal{L}_{\varepsilon\Sigma}(u) = f_{neck}$ where $\frac{1}{4}\mathcal{L}_{\varepsilon\Sigma}$ is the linearized mean curvature operator of $\varepsilon\Sigma$ with this metric. In addition, $f_{neck,\rho}$ is invariant under the symmetries of $\varepsilon\Sigma$ induced by the transformations $(y^1, \hat{y}) \longmapsto (-y^1, \hat{y})$ and $(y^1, \hat{y}) \longmapsto (y^1, B\hat{y})$ for all $B \in O(n)$.

Parametrize the catenoid by (s, Θ) as in Definition 3 and let $|\cdot|_{C^{k,\alpha}_\delta(\varepsilon_a K)}$ denote the standard weighted $C^{l,\beta}_\delta$ norm on $\varepsilon\Sigma$, so that

$$|u|_{C^{l,\beta}_\delta(\varepsilon\Sigma)} := |(\varepsilon \cosh(s))^{-\delta} u|_{0,\varepsilon\Sigma} + \cdots + |(\varepsilon \cosh(s))^{-\delta+l} \nabla^l u|_{0,\varepsilon\Sigma}$$
$$+ [(\varepsilon \cosh(s))^{-\delta+l+\beta} \nabla^l u]_{\beta,\varepsilon\Sigma}$$

where the norms and derivatives correspond to the metric on $\varepsilon\Sigma$. In this parametrization, the symmetries induced on functions of $\varepsilon\Sigma$ are $u(s, \Theta) = u(-s, \Theta)$ and $u(s, \Theta) = u(s, B(\Theta))$ for all $B \in O(n)$ (which just says that u is independent of Θ). The corresponding spaces of functions invariant under these symmetries will be denoted by $C^{l,\beta}_{\delta,sym}(\varepsilon\Sigma)$.

A standard separation of variables argument shows that when $\delta < 0$, the kernel of the operator $\frac{1}{4}\mathcal{L}_{\varepsilon\Sigma} : C^{2,\beta}_\delta(\varepsilon\Sigma) \to C^{0,\beta}_{\delta-2}(\varepsilon\Sigma)$ consists of the linear span of the Jacobi fields $\{J_k : k = 2, \ldots, n+1\}$. However, none of these Jacobi fields is invariant under the symmetries $(s, \Theta) \longmapsto (s, B(\Theta))$ for $B \in O(n)$. Hence the operator $\frac{1}{4}\mathcal{L}_{\varepsilon\Sigma} : C^{2,\beta}_{\delta,sym}(\varepsilon\Sigma) \to C^{0,\beta}_{\delta-2,sym}(\varepsilon\Sigma)$ is injective for $\delta < 0$; and by duality it is surjective for $\delta > 2 - n$. Let $u_{neck} \in C^{2,\beta}(\varepsilon\Sigma)$ be the unique solution of the equation $\frac{1}{4}\mathcal{L}_{\varepsilon\Sigma}(u_{neck}) = f_{neck}$. The estimate $|u_{neck}|_{C^{2,\beta}_\delta(\varepsilon\Sigma)} \leq C|f_{neck}|_{C^{0,\beta}_{\delta-2}(\varepsilon\Sigma)} \leq C|f|_{C^{0,\beta}_{\delta-2}}$ is valid, where C is a constant independent of ε. With slight abuse of notation, extend this function to all of $\tilde{\Lambda}_{\alpha,\tau}$ by defining $\bar{u}_{neck} := \sum_{k=0}^{N-1} \chi^k_{neck,\rho} \cdot u_{neck}$. One has the estimate $|\bar{u}_{neck}|_{C^{2,\beta}_\delta} \leq C|f|_{C^{0,\beta}_{\delta-2}}$.

Step 2. Local solutions on the exterior regions. Given the local solution $\bar{u}_{neck}$ constructed in the previous step, choose a small $\kappa \in (0, 1)$ and define

$$\hat{f}_{ext}^k := \chi_{ext,\kappa\rho}^k \big(f - \tilde{\mathcal{L}}_{\alpha,\tau}(\bar{u}_{neck}) \big).$$

This function vanishes within an ε-independent distance from the union of all the neck regions of $\tilde{\Lambda}_{\alpha,\tau}$. Therefore one can determine without difficulty $|\hat{f}_{ext}^k|_{C^{0,\beta}} \leq C_\kappa |f|_{C^{0,\beta}_{\delta-2}}$ for some constant C_κ that depends on κ and δ. Here, $|\cdot|_{C^{0,\beta}}$ is the un-weighted Schauder norm.

By symmetry, one can say that $\hat{f}_{ext}^k := \hat{f}_{ext}$ for each k. In addition, $\hat{f}_{ext}$ is invariant under the symmetries T and S_B^{01} for all $B \in O(n)$. Denote the space of $C^{l,\beta}$ functions on S_α possessing these symmetries by $C_{sym}^{l,\beta}(S_\alpha)$. The function $\hat{f}_{ext}$ can be viewed as a function of compact support on the perturbed hypersphere $\tilde{S}_\alpha$. Since $\tilde{S}_\alpha$ is a normal graph over the hypersphere S_α, this function can be pulled back to the hypersphere S_α. The metric carried by S_α in this identification is a perturbation of the standard induced metric $\sin^2(\alpha)g_{\mathbb{S}^n}$. However, the equation that will be solved here is $\mathcal{L}_\alpha(u) = \hat{f}_{ext}$ where $\mathcal{L}_\alpha$ is the linearized mean curvature operator of S_α when it carries the unperturbed metric $\sin^2(\alpha)g_{\mathbb{S}^n}$.

By another standard argument, one can show that the kernel of $\mathcal{L}_\alpha$ on S_α consists of the linear span of the coordinate functions x^k restricted to S_α. None of these functions satisfies all the required symmetries; hence the operator L_α has no kernel when the symmetry conditions are imposed and thus $\mathcal{L}_\alpha : C_{sym}^{2,\beta}(S_\alpha) \to C_{sym}^{0,\beta}(S_\alpha)$ is bijective. Let $u_{ext} \in C_{sym}^{2,\beta}(S_\alpha)$ denote the unique solution of the equation $\mathcal{L}_\alpha(u_{ext}) = \hat{f}_{ext}$. One has the estimate $|u_{ext}|_{C^{2,\beta}(S_\alpha)} \leq C|\hat{f}_{ext}|_{C^{0,\beta}} \leq C_\kappa |f|_{C^{0,\beta}_{\delta-2}}$.

To proceed, recall the $(\mu, \Theta) \in (0, \pi) \times \mathbb{S}^{n-1}$ parametrization for $S_\alpha \setminus \{p^+, p^-\}$. By examining the Taylor expansion of u_{ext} at the points $\mu_0 = 0$ or $\mu_0 = \pi$ and invoking its symmetries, one finds that $u_{ext} = a + v_{ext}$ where $a := u_{ext}(\mu_0)$, while $|v_{ext}(\mu)| \leq C_\kappa |\mu - \mu_0|^2 |f|_{C^{0,\beta}_{\delta-2}}$. One can now extend u_{ext} to all of $\tilde{\Lambda}_{\alpha,\tau}$ as follows. Recall that J_0 is the bounded Jacobi field on the neck having even symmetry, which can be normalized to have the asymptotic expansion $J_0(s) = 1 + \tilde{J}_0(s)$ where $\tilde{J}_0(s) = \mathcal{O}(\cosh^{2-n}(s))$ for large $|s|$. Now define, again with slight abuse of notation, the function

$$\bar{u}_{ext} := \sum_{k=0}^{N-1} \eta_{ext}^k \cdot u_{ext} + \sum_{k=0}^{N-1} \eta_{neck}^k \cdot a J_0.$$

Step 3. Estimates and convergence. In the preceding two steps, local solutions $\bar{u}_{neck}$ and $\bar{u}_{ext}$ on the neck regions and on the exterior regions, respectively, were found and extended to all of $\tilde{\Lambda}_{\alpha,\tau}$. Define $\bar{u} := \bar{u}_{neck} + \bar{u}_{ext}$. Then

$$\begin{aligned}
\tilde{\mathcal{L}}_{\alpha,\tau}(\bar{u}) - f = \sum_{k=0}^{N-1} \Big[&\eta_{ext}^k (\tilde{\mathcal{L}}_{\alpha,\tau} - \mathcal{L}_\alpha)(u_{ext}) + \chi_{neck,\kappa\rho}^k (\tilde{\mathcal{L}}_{\alpha,\tau} - \tfrac{1}{4}\mathcal{L}_{\varepsilon\Sigma})(u_{neck}) \\
&+ \eta_{neck}^k (\tilde{\mathcal{L}}_{\alpha,\tau} - \tfrac{1}{4}\mathcal{L}_{\varepsilon\Sigma})(a J_0) + [\tilde{\mathcal{L}}_{\alpha,\tau}, \eta_{neck}^k](a \tilde{J}_0) + [\tilde{\mathcal{L}}_{\alpha,\tau}, \eta_{ext}^k](v_{ext}) \Big]
\end{aligned}$$

$$(22)$$

where $[\mathcal{L}, \eta](u) := \mathcal{L}(\eta u) - \eta \mathcal{L}(u)$. Each term in (22) will be shown to be small in the $C_{\delta-2}^{0,\beta}$ norm.

Begin with the first term in (22). Note that this term is supported in $\mathcal{E}_\varepsilon$ meaning that it vanishes within a distance of $\rho_\varepsilon = \varepsilon^{(3n-3)/(3n-2)}$ from the centres of the neck regions.

The fact that the operator $\tilde{\mathcal{L}}_{\alpha,\tau}$ differs only very slightly from the operator $\mathcal{L}_{\alpha}$ away from the neck regions is the key to deriving the estimate. That is, because $\tilde{\Lambda}_{\alpha,\tau}$ is the normal graph over S_{α} of the function $\alpha + \varepsilon^{n-1}G$ defined in Sect. 2.3, one can use Lemma 2 to find the form of $\tilde{\mathcal{L}}_{\alpha,\tau}$. This yields a rather unwieldy expression; however, with some work one finds

$$
\begin{aligned}
(\tilde{\mathcal{L}}_{\alpha,\tau} - \mathcal{L}_{\alpha})(u) &= (\tilde{\Delta} - \Delta_{\alpha})(u) + (\|\tilde{B}\|^2 - \|B_{\alpha}\|^2)u \\
&= \frac{\Delta u + \mathcal{O}(\varepsilon^{2n-2}\|\nabla^2 G\|\|\nabla G\|) \cdot \nabla u + \mathcal{O}(\varepsilon^{2n-2}\|\nabla G\|^2) \cdot \nabla^2 u}{\sin^2(\alpha + \varepsilon^{n-1}G)} - \frac{\Delta u}{\sin^2(\alpha)} \\
&\quad + \left(\frac{n \sin^4(\alpha + \varepsilon^{n-1}G)\cos^2(\alpha + \varepsilon^{n-1}G) + \mathcal{O}(\varepsilon^{2n-2}\|\nabla^2 G\|^2)}{(\sin^2(\alpha + \varepsilon^{n-1}G) + \varepsilon^{2n-2}\|\nabla G\|^2)^3} - n\cot^2(\alpha) \right)u \\
&= \mathcal{O}(\varepsilon^{2n-2}\|\nabla^2 G\|\|\nabla G\|) \cdot \nabla u + \left(\mathcal{O}(\varepsilon^{n-1}G) + \mathcal{O}(\varepsilon^{2n-2}\|\nabla G\|^2) \right) \cdot \nabla^2 u \\
&\quad + \left(\mathcal{O}(\varepsilon^{n-1}G) + \mathcal{O}(\varepsilon^{2n-2}\|\nabla^2 G\|^2) \right)u
\end{aligned}
\tag{23}
$$

evaluated on an arbitrary function $u \in C_{\delta}^{2,\beta}(\tilde{\Lambda}_{\alpha,\tau})$, where $\tilde{\Delta}$ is the Laplacian of $\tilde{\Lambda}_{\alpha,\tau}$ and $\tilde{B}$ is its mean curvature. The notation $\mathcal{O}(*) \cdot \nabla^k u$ means a linear quantity in $\nabla^k u$ with coefficients satisfying $\mathcal{O}(*)$. The reason one reaches (23) is because the largest terms of $\tilde{\Delta} - \Delta_{\alpha}$ and $\|\tilde{B}\|^2 - \|B_{\alpha}\|^2$, as computed from Lemma 2, have been retained at the expense of all the smaller terms. By recalling the expansion of G given in Eq. 5, one can now compute

$$
|(\tilde{\mathcal{L}}_{\alpha,\tau} - \mathcal{L}_{\alpha})(u_{ext})|_{C_{\delta-2}^0(\mathcal{E}_\varepsilon)} \le C\varepsilon^{(4n-3)/(3n-2)}|u_{ext}|_{C_{\delta}^{2,\beta}(\mathcal{E}_\varepsilon)} \le C\varepsilon^{(4n-3)/(3n-2)}|f|_{C_{\delta-2}^{0,\beta}}.
$$

An estimate for the Hölder coefficient can be found by similar computations.

Next, using the fact that the operator $\tilde{\mathcal{L}}_{\alpha,\tau}$ differs only very slightly from the operator $\mathcal{L}_{\varepsilon\Sigma}$ near the neck regions, one can estimate the second term in (22). Begin with the computation in $\mathcal{N}_{\varepsilon}^k$. Here, one employs Lemma 4 and the parametrization for the catenoid used in Definition 3 to calculate

$$
\tilde{\mathcal{L}}_{\alpha,\tau}(u) = A^2 \mathcal{L}_{\varepsilon\Sigma}(u) + (2-n)A\left(\frac{\psi\dot{\psi} + \phi\dot{\phi}}{\phi^2} \right)\frac{\partial u}{\partial s} + n\left(1 + \varepsilon^2\left(\frac{\dot{\phi}\dot{\psi}}{\phi} - \frac{1}{\phi^{n-2}} \right) \right)u
\tag{24}
$$

where $A = \frac{1}{2}\left(1 + \varepsilon^2(\psi^2 + \phi^2) \right)$. Therefore

$$
\begin{aligned}
|(\tilde{\mathcal{L}}_{\alpha,\tau} - \tfrac{1}{4}\mathcal{L}_{\varepsilon\Sigma})(u_{neck})|_{C_{\delta-2}^0(\mathcal{N}_{\varepsilon}^k)} &= \mathcal{O}(\varepsilon^2\cosh^2(s))\nabla^2 u_{neck} + \mathcal{O}(\varepsilon\cosh(s))\nabla u_{neck} \\
&\quad + \mathcal{O}(1)u_{neck} \le C\varepsilon^{2(3n-3)/(3n-2)}|u_{neck}|_{C_{\delta}^{2,\beta}(\mathcal{N}_{\varepsilon}^k)} \\
&\le C\varepsilon^{2(3n-3)/(3n-2)}|f|_{C_{\delta-2}^{0,\beta}}.
\end{aligned}
$$

Now consider the same quantity further away from the neck region, in the region $\mathcal{N}_{\varepsilon}^k(2\kappa\rho)$. Similar calculations can be performed, this time computing $\tilde{\mathcal{L}}_{\alpha,\tau}$ from Lemma 4 with a parametrization of a small perturbation of the catenoid. The perturbation introduces additional error terms, but the largest error still comes from the $\mathcal{O}(1)$ terms. Hence one obtains the estimate

$$
|(\tilde{\mathcal{L}}_{\alpha,\tau} - \tfrac{1}{4}\mathcal{L}_{\varepsilon\Sigma})(u_{neck})|_{C_{2-\delta}^0(\mathcal{N}_{\varepsilon}^k(2\kappa\rho)\setminus\mathcal{N}_{\varepsilon}^k)} \le C\kappa^2|u_{neck}|_{C_{\delta}^{2,\beta}(\mathcal{N}_{\varepsilon}^k(2\kappa\rho)\setminus\mathcal{N}_{\varepsilon}^k)} \le C\kappa^2|f|_{C_{\delta-2}^{0,\beta}}.
$$

Analogous Hölder coefficient estimates can be found in the same way. In the end, one finds

$$|\chi_{neck,\kappa\rho}^k(\tilde{\mathcal{L}}_{\alpha,\tau} - \tfrac{1}{4}\mathcal{L}_{\varepsilon\Sigma})(u_{neck})|_{C_{\delta-2}^{0,\beta}} \leq C\big(\kappa^2 + \varepsilon^{2(3n-3)/(3n-2)}\big)|f|_{C_{\delta-2}^{0,\beta}}. \qquad (25)$$

The remaining terms in (22) are considerably easier to estimate. The third term can be estimated in much the same way as the first. In fact, formula (24) applied to the function $u = J_0$ and the estimate $|a| \leq C|u_{ext}|_{C^{2,\beta}} \leq C_\kappa|f|_{C_{\delta-2}^{0,\beta}}$ gives

$$|a(\tilde{\mathcal{L}}_{\alpha,\tau} - \tfrac{1}{4}\mathcal{L}_{\varepsilon\Sigma})(\eta_{neck}^k J_0)|_{C_\delta^{2,\beta}} \leq C_\kappa \varepsilon^{2(3n-2)/(3n-2)}|f|_{C_{\delta-2}^{0,\beta}}.$$

Finally, the estimate of the last term follows from the fact that in the support of the $[\tilde{\mathcal{L}}_{\alpha,\tau}, \eta]$ quantities, the functions v_{ext} and $a\tilde{J}_0$ are of size $\mathcal{O}([\mathrm{dist}(\gamma, \cdot)]^2|f|_{C_{\delta-2}^{0,\beta}})$ while the $C_{\delta-2}^{0,\beta}$ norm of the coefficients of $[\tilde{\mathcal{L}}_{\alpha,\tau}, \eta]$ are $\mathcal{O}(1)$ by definition of the cut-off functions. Hence an estimate proportional to $|f|_{C_{\delta-2}^{0,\beta}}$ is again valid, where the proportionality constant is of size $\mathcal{O}(\varepsilon^{2(3n-2)/(3n-3)})$.

The outcome of the above analysis is that one has a function $\bar{u} \in C_\delta^{2,\beta}(\tilde{\Lambda}_{\alpha,\tau})$ satisfying the estimates $|\bar{u}|_{C_\delta^{2,\beta}} \leq C_\kappa|f|_{C_{\delta-2}^{0,\beta}}$ as well as

$$|\tilde{\mathcal{L}}_{\alpha,\tau}(\bar{u}) - f|_{C_{\delta-2}^{0,\beta}} \leq C\big(\kappa^2 + \varepsilon^{2(3n-3)/(3n-2)} + \varepsilon^{(4n-3)/(3n-2)}\big)|f|_{C_{\delta-2}^{0,\beta}}$$

where C is bounded above independently of κ. Since the constant in front of $|f|_{C_{\delta-2}^{0,\beta}}$ can be made as small as desired by appropriate choices of ε and κ, a straightforward iteration argument now proves the existence of an actual solution $u \in C_\delta^{2,\beta}(\tilde{\Lambda}_{\alpha,\tau})$ of the equation $\tilde{\mathcal{L}}_{\alpha,\tau}(u) = f$ satisfying the estimate $|u|_{C_\delta^{2,\beta}} \leq C|f|_{C_{\delta-2}^{0,\beta}}$. $\square$

A modification of the preceding proof, which takes into account the double indicial root of $\tilde{\mathcal{L}}_{\alpha,\tau}$, can be found to handle the two-dimensional case.

Proposition 13 *Suppose $n = 2$ and choose $\delta \in (-1, 0)$ and $\tau > 0$ sufficiently small. The linearized operator $\tilde{\mathcal{L}}_{\alpha,\tau} : C_\delta^{2,\beta}(X) \to C_{\delta-2}^{0,\beta}(X)$ satisfies the estimate*

$$|\tilde{\mathcal{L}}_{\alpha,\tau}(u)|_{C_{\delta-2}^{0,\beta}} \geq C\varepsilon^{-\delta}|u|_{C_\delta^{2,\beta}}$$

where $\varepsilon = \varepsilon(\tau)$ is the scale parameter of $\tilde{\Lambda}_{\alpha,\tau}$ and C is a constant independent of τ.

Proof The proof of the result in dimension $n = 2$ follows broadly the same plan as the proof in higher dimensions. For given $f \in C_{\delta-2}^{0,\beta}(\tilde{\Lambda}_{\alpha,\tau})$, a solution of the equation $\tilde{\mathcal{L}}_{\alpha,\tau}(u) = f$ must be found and estimated. Begin by defining the different regions of $\tilde{\Lambda}_{\alpha,\tau}$ and their corresponding cut-off functions exactly as before, along with the decomposition $f = \sum_{k=0}^{N-1}(f_{neck}^k + f_{ext}^k)$.

The significant difference between this proof and the preceding one is how the local solutions are found on the neck. The reason is that due to the indicial roots, there is no range of δ in which the operator $\mathcal{L}_{\varepsilon\Sigma} : C_{\delta,sym}^{2,\beta}(\varepsilon\Sigma) \to C_{\delta-2,sym}^{0,\beta}(\varepsilon\Sigma)$ is both injective and surjective. One must thus proceed differently.

First, recall that in dimension $n = 2$, the function J_0 is the linearly growing Jacobi field on the neck having even symmetry, which can be normalized to have the asymptotic expansion

$J_0(s) = -1 + |s| + \tilde{J}_0(s)$ where $\tilde{J}_0(s) = \mathcal{O}(\cosh^{-2}(s))$ for large $|s|$. The bounded Jacobi field having odd symmetry is J_1 and can be normalized to have the asymptotic expansion $J_1(s) = 1 + \tilde{J}_1(s)$ where $\tilde{J}_1(s) = \mathcal{O}(\cosh^{-2}(s))$ for large positive s. Define a smooth, odd function $\chi : \varepsilon\Sigma \to \mathbb{R}$ with the property $\chi(s) = 1$ for $s \geq 1$ and $\chi(s) = -1$ for $s \leq -1$. Now set $K_1 := \chi J_1$ and define the linear subspaces $\mathcal{D} := \mathrm{span}_{\mathbb{R}}\{J_0, K_1\}$. Together, J_0 and K_1 span all possible asymptotic behaviours corresponding to the indicial root at zero of $\mathcal{L}_{\varepsilon\Sigma}$. The following two facts hold, and are well known in the theory of elliptic operators on asymptotically flat manifolds. The operator $\mathcal{L}_{\varepsilon\Sigma} : C^{2,\beta}_{\delta,sym}(\varepsilon\Sigma) \oplus \mathcal{D} \to C^{0,\beta}_{\delta-2,sym}$ is surjective in the range $\delta \in (-1, 0)$ with one-dimensional kernel spanned by J_0. Furthermore, there is a bounded right inverse mapping into $C^{2,\beta}_{\delta,sym}(\varepsilon\Sigma) \oplus \mathcal{D}_0$ where $\mathcal{D}_0 := \mathrm{span}_{\mathbb{R}}\{K_1\}$. The proof of these facts follows from a similar proof that can be found, for example, in [18] in the context of CMC surfaces. Consequently, there is a solution $u_{neck} \in C^{2,\beta}_{\delta,sym}(\varepsilon\Sigma) \oplus \mathcal{D}_0$ of the equation $\mathcal{L}_{\varepsilon\Sigma}(u_{neck}) = f_{neck})$ which can be decomposed as $u_{neck} = v_{neck} + a_1 K_1$ where $v_{neck} \in C^{2,\beta}_{\delta,sym}(\varepsilon\Sigma)$ and $a_1 \in \mathbb{R}$ satisfy

$$|v_{neck}|_{C^{2,\beta}_{\delta}(\varepsilon\Sigma)} + \varepsilon^{-\delta}|a_1| \leq C|f_{neck}|_{C^{0,\beta}_{\delta-2}(\varepsilon\Sigma)} \leq C|f|_{C^{0,\beta}_{\delta-2}}$$

for some constant C independent of ε.

Recall now that G is the singular solution of $\mathcal{L}_\alpha(G) = 0$ on $S_\alpha \setminus \{p^+, p^-\}$, for which one has $G(s) = \gamma_0 + \gamma_1|s| + \tilde{G}(s)$ where $\tilde{G}(s) = \mathcal{O}(\varepsilon^2 \cosh^2(s))$ and γ_0 and γ_1 are non-zero constants. Note that $\gamma_0 = O(|\log(\varepsilon)|)$. This follows from the expansion for G and a formula for changing between the μ-coordinate of S_α and the s-coordinate of $\varepsilon\Sigma$ on the regions of overlap. The function u_{neck} can thus be extended to $\tilde{\Lambda}_{\alpha,\tau}$ by prescribing

$$\bar{u}_{neck} := \sum_{k=0}^{N-1} \chi^k_{neck,\rho} v_{neck} + \sum_{k=0}^{N-1} \left(\eta^k_{neck}(b_1 J_0 + a_1 K_1) + \eta^k_{ext} c_1 G \right)$$

where the constants b_1 and c_1 are chosen to ensure matching in the constant and linear terms of asymptotic expansions of u_{neck} and G, i.e., $b_1 = -a_1\gamma_1/(\gamma_1+\gamma_2)$ and $c_1 = -a_1/(\gamma_0+\gamma_1)$. In the end, this solution satisfies the estimates $|\bar{u}_{neck}|_{C^{2,\beta}_\delta} \leq C\varepsilon^\delta|f|_{C^{0,\beta}_{\delta-2}}$ as well as

$$|\chi^k_{neck,\kappa\rho}(\tilde{\mathcal{L}}_{\alpha,\tau}(\bar{u}_{neck}) - f_{neck})|_{C^{0,\beta}_{\delta-2}} \leq C\left(\kappa^2 + \varepsilon^{3/2+\delta}|\log(\varepsilon)|\right)|f|_{C^{0,\beta}_{\delta-2}}.$$

Note that $\delta > -3/2$ is required to ensure that the quantity in front of $|f|_{C^{0,\beta}_{\delta-2}}$ small as $\varepsilon \to 0$.

Now that one has the solution $\bar{u}_{neck}$ on the neck region and extended to all of $\tilde{\Lambda}_{\alpha,\tau}$, the next step is to define $\hat{f}_{ext}$ in the same way as before and find the local solution on the exterior region of $\tilde{\Lambda}_{\alpha,\tau}$. The extension to all of $\tilde{\Lambda}_{\alpha,\tau}$ is again complicated by the fact that one must match constant and linear terms in the Taylor series at the points p^+ and p^-. This can be accomplished by using G and J_0. That is, define the extended function

$$\bar{u}_{ext} := \sum_{k=0}^{N-1} \eta^k_{ext} \cdot (u_{ext} + cG) + \sum_{k=0}^{N-1} \eta^k_{neck} \cdot b J_0$$

where b and c are chosen to ensure matching in the constant and linear terms of the Taylor expansion; i.e., $b = -a\gamma_1/(\gamma_1 + \gamma_2)$ and $c = -a/(\gamma_0 + \gamma_1)$, where $a = u_{ext}(0)$.

The remainder of the analysis is the same as before, and leads to a good approximate solution of the equation $\tilde{\mathcal{L}}_{\alpha,\tau}(u) = f$. The solution procedure can be iterated to yield an exact solution. The result is a solution satisfying the bound $|u|_{C^{2,\beta}_\delta} \leq C\varepsilon^\delta|f|_{C^{0,\beta}_{\delta-2}}$. □

2.8 The non-linear estimates and the conclusion of the proof

2.8.1 The non-linear estimates

According to the discussion in Sect. 2.5, the proof of Main Theorem 1 requires two more estimates in addition to the linear estimate from the previous section: it is necessary to show that $\Phi_{\alpha,\tau}(0)$ is small in the $C^{0,\beta}_{\delta-2}$ norm; and it is necessary to show that $D\Phi_{\alpha,\tau}(f) - \tilde{\mathcal{L}}_{\alpha,\tau}$ can be made to have small $C^{2,\beta}_{\delta}$-operator norm if f is chosen sufficiently small in the $C^{2,\beta}_{\delta}$ norm. These two estimates are the subject of this section. Once these estimates are given, it will be possible to conclude the proof of Main Theorem 1 by invoking the inverse function theorem. Begin with the first non-linear estimate.

Proposition 14 *The quantity* $\Phi_{\alpha,\tau}(0)$, *which is the mean curvature of* $\tilde{\Lambda}_{\alpha,\tau}$, *satisfies the following estimate. If* $\tau > 0$ *is sufficiently small, then there exists a constant C independent of τ so that*

$$|\Phi_{\alpha,\tau}(0)|_{C^{0,\beta}_{\delta-2}} \leq C\varepsilon^{(2-\delta)(3n-3)/(3n-2)} \tag{26}$$

where $\varepsilon = \varepsilon(\tau)$ *is the scale parameter of* $\tilde{\Lambda}_{\alpha,\tau}$ *as in Sect. 2.3.*

Proof The estimate (26) is the consequence of three separate calculations: in $\mathcal{E}_\varepsilon$ where $|\zeta_\varepsilon| \geq 2\rho_\varepsilon$, in $\mathcal{T}_\varepsilon$ where $|\zeta_\varepsilon| = \mathcal{O}(\rho_\varepsilon)$, and in $\mathcal{N}_\varepsilon$ where $|\zeta_\varepsilon| \leq \rho_\varepsilon/2$. For the first of these, one uses the expression for the mean curvature of a normal graph over the hypersphere S_α from Lemma 2 for the graphing function $F = \varepsilon^{n-1}G$. One obtains the expression

$$H\big(\exp(\varepsilon^{n-1}GN_\alpha)(S_\alpha)\big) = \frac{-\varepsilon^{n-1}\Delta G + n\sin(\alpha + \varepsilon^{n-1}G)\cos(\alpha + \varepsilon^{n-1}G)}{A\sin(\alpha + \varepsilon^{n-1}G)}$$
$$-\frac{\varepsilon^{2n-2}\big[\varepsilon^{n-1}\ddot{G} + \cos(\alpha + \varepsilon^{n-1}G)\sin(\alpha + \varepsilon^{n-1}G)\big]\dot{G}^2}{A^3\sin(\alpha + \varepsilon^{n-1}G)} \tag{27}$$

where Δ is the Laplacian of the standard metric of $\mathbb{S}^n$ and $A = \big(\sin^2(\alpha + \varepsilon^{n-1}G) + \varepsilon^{2n-2}\dot{G}^2\big)^{1/2}$. Here G is the singular solution of $(\Delta + n)(G) = 0$ and ε is very small. Thus expanding (27) yields

$$H\big(\exp(\varepsilon^{n-1}G)(S_\alpha)\big) = n\cot(\alpha) - \varepsilon^{2n-2}\big((n-2)\dot{G}^2 + 2nG^2\big)\cot(\alpha)\csc^2(\alpha)$$
$$+ \varepsilon^{3n-3}\mathcal{O}\big(|\ddot{G}|\dot{G}^2 + |G|\,\dot{G}^2 + |G|^3\big).$$

Since $H_\alpha = n\cot(\alpha)$ and using the expansion of G given in (5) as well as the fact that $\mathrm{dist}(\gamma, q) \geq \rho_\varepsilon$ in $\mathcal{E}_\varepsilon$, then one finds

$$|H(\tilde{\Lambda}_{\alpha,\tau}) - H_\alpha|_{C^0_{2-\delta}(\mathcal{E}_\varepsilon)} \leq C\varepsilon^{(2-\delta)(3n-3)/(3n-2)}.$$

This completes the supremum estimate needed for the $C^{0,\beta}_{\delta-2}$ estimate in $\mathcal{E}_\varepsilon$. The Hölder coefficient estimate needed for the $C^{0,\beta}_{\delta-2}$ estimate can be found in a similar (though more involved) way.

The remaining two calculations use Lemma 4, namely, the formula for the mean curvature of an embedded hypersurface in $\mathbb{R}^{n+1}$ with respect to the metric induced from $\mathbb{S}^{n+1}$ by stereographic projection. Consider first the neck region $\mathcal{N}_\varepsilon$ whose image under stereographic projection is a catenoid with vanishing mean curvature with respect to the Euclidean metric.

Therefore by Lemma 4, the mean curvature at a point $(\varepsilon\psi(s), \varepsilon\phi(s)\Theta) \in \varepsilon\Sigma \cap \{(y^1, \hat{y}) : \|\hat{y}\| \leq \rho_\varepsilon\}$ is

$$H(\tilde{\Lambda}_{\alpha,\tau})(s) = n\varepsilon \frac{\dot{\phi}(s)\psi(s) - \dot{\psi}(s)\phi(s)}{(\dot{\psi}(s)^2 + \dot{\phi}(s)^2)^{1/2}} = \begin{cases} \mathcal{O}(\varepsilon|\log(\varepsilon)|) & n = 2 \\ \mathcal{O}(\varepsilon) & n \geq 3. \end{cases}$$

This is because $\phi(s) \in [\varepsilon, \rho_\varepsilon]$, while $\dot{\phi}(s) = \mathcal{O}(\phi(s))$ and $\psi(s) = c(\varepsilon, n) + \mathcal{O}(\phi(s)^{2-n})$ for large s, where $c(\varepsilon, 2) = \mathcal{O}(|\log(\varepsilon)|)$ and $c(\varepsilon, n) = \mathcal{O}(1)$ for $n \geq 2$. Thus

$$|H(\tilde{\Lambda}_{\alpha,\tau}) - H_\alpha|_{C^0_{2-\delta}(\mathcal{N}_\varepsilon)} \leq C\varepsilon^{(2-\delta)(3n-3)/(3n-2)}.$$

Once again, the Hölder coefficient is similar, and yields the same sort of bound. Finally, it remains to estimate the mean curvature in the region $\mathcal{T}_\varepsilon$ whose image under stereographic projection is the graph of the function $F := \tilde{F}_{a,\tau}$ defined in Eq. 15. Let $y = \|\hat{y}\|$. Then at the point $(F(y), \hat{y}) \in \tilde{\Sigma}_\varepsilon \cap \{(y^1, \hat{y}) : \rho_\varepsilon/2 \leq y \leq 2\rho_\varepsilon\}$, the formula of Lemma 4 yields

$$H(\tilde{\Lambda}_{\alpha,\tau})(y) = \frac{1 + y^2 + F(y)^2}{2}\left(\frac{\ddot{F}(y)}{(1 + \dot{F}(y)^2)^{3/2}} + \frac{(n-1)\dot{F}(y)}{y(1 + \dot{F}(y)^2)^{1/2}}\right)$$
$$+ \frac{n(F(y) - y\dot{F}(y))}{(1 + \dot{F}(y)^2)^{1/2}}.$$

The function $F(y)$ has the form

$$F(y) = \varepsilon c_n + C_n \varepsilon^{n-1} L_n(y) + \eta\left(\mathcal{O}(y^2) + \mathcal{O}(\varepsilon^{3n-3}y^{4-3n})\right)$$

where $L_n(y) = \log(y)$ when $n = 2$ and $L_n(y) = y^{2-n}$ when $n \geq 3$, while C_n and c_n are constants and η is the cut-off function whose $C^{2,\beta}_0$ norm is bounded by an ε-independent constant. Using the fact that L_n satisfies the zero mean curvature equation $\ddot{L}_n + \frac{n-1}{y}(1 + \dot{L}_n^2)\dot{L}_n = 0$ then one sees that

$$|H(\tilde{\Lambda}_{\alpha,\tau}) - H_\alpha|_{C^0_{2-\delta}(\mathcal{T}_\varepsilon)} \leq C\varepsilon^{(2-\delta)(3n-3)/(3n-2)}$$

in the region $y \in [\rho_\varepsilon/2, 2\rho_\varepsilon]$. Once again, the Hölder coefficient estimate in this region is similar, and yields the same sort of bound. □

The second non-linear estimate is essentially the same as [5, Prop. 26] and [6, Prop. 28]. The result is re-stated here for reference in the proof of Main Theorem 1 that will follow below.

Proposition 15 *The linearized mean curvature operator satisfies the following general estimate. If $\varepsilon > 0$ is sufficiently small and $f \in C^{2,\beta}_\delta(\tilde{\Lambda}_{\alpha,\tau})$ has sufficiently small $C^{2,\beta}_\delta$ norm, then there exists a constant C independent of ε so that*

$$\left|\mathrm{D}\Phi_{\alpha,\tau}(f)(u) - \mathcal{L}_{\alpha,\tau}(u)\right|_{C^{0,\beta}_{\delta-2}} \leq C\varepsilon^{\delta-1}|f|_{C^{2,\beta}_\delta}|u|_{C^{2,\beta}_\delta} \tag{28}$$

for any function $u \in C^{2,\beta}_\delta(\tilde{\Lambda}_{\alpha,\tau})$.

Proof This holds because of the natural scaling property of the mean curvature operator. Pick a point $q \in \tilde{\Lambda}_{\alpha,\tau}$ and work in geodesic normal coordinates in a neighbourhood $\mathcal{U}$ of q. Let $\mathcal{U}_f$ denote the normal deformation of $\mathcal{U}$ determined by f. Now set $s := \zeta_\varepsilon(q)$ and consider re-scaling by the factor $1/s$. The mean curvature at q satisfies $H_1(\mathcal{U}_f) = s^{-1}H_{1/s}(s^{-1}\mathcal{U}_{f/s})$ where H_r is the mean curvature of a hypersurface of the sphere of radius r. Hence

$$s^{2-\delta}|\mathrm{D}H_1(\mathcal{U}_f)(u) - \mathrm{D}H_1(\mathcal{U}_0)(u)| = s^{1-\delta}|\mathrm{D}H_{1/s}(s^{-1}\mathcal{U}_{f/s})(u) - \mathrm{D}H_{1/s}(s^{-1}\mathcal{U}_0)(u)|$$

$$\leq C s^{-1-\delta} |f|_{C^{2,\beta}(s^{-1}\mathcal{U}_0)} |u|_{C^{2,\beta}(s^{-1}\mathcal{U}_0)}$$

$$\leq C s^{\delta-1} |f|_{C^{2,\beta}_\delta(\mathcal{U}_0)} |u|_{C^{2,\beta}_\delta(\mathcal{U}_0)} \tag{29}$$

by performing the scaling backwards. Here C is a universal constant independent of s. Allowing q to range over all the points in $\tilde{\Lambda}_{\alpha,\tau}$ now yields the desired estimate; the constant appearing in (28) arises because the largest $s^{-1+\delta}$ can be in the right-hand side of (29) is $\mathcal{O}(\varepsilon^{\delta-1})$. □

The proof of Main Theorem 1. The estimates for the proof of Main Theorem 1 are now all in place and the conclusion of the theorem becomes a simple verification of the conditions of the inverse function theorem. By Propositions 12 and 13 the linearization $\tilde{\mathcal{L}}_{\alpha,\tau}$ is injective on $C^{2,\beta}_\delta(X)$. But $\tilde{\mathcal{L}}_{\alpha,\tau} - \Delta$ is a compact operator so $\tilde{\mathcal{L}}_{\alpha,\tau}$ has the same index as Δ on $C^{2,\beta}_\delta(X)$. By self-adjointness, this index is zero, so that $\tilde{\mathcal{L}}_{\alpha,\tau}$ must be surjective as well. One has the estimate

$$|\tilde{\mathcal{L}}^{-1}_{\alpha,\tau}(f)|_{C^{2,\beta}_\delta} \leq C_L |f|_{C^{0,\beta}_{\delta-2}},$$

where $C_L = \mathcal{O}(\varepsilon^\delta)$ in dimension $n = 2$ and $C_L = \mathcal{O}(1)$ in higher dimensions. Now in order to achieve the bound

$$|\mathrm{D}\Phi_{\alpha,\tau}(f)(u) - \tilde{\mathcal{L}}_{\alpha,\tau}(u)|_{C^{0,\beta}_{\delta-2}} \leq \frac{1}{2C_L} |u|_{C^{2,\beta}_\delta},$$

for any $u \in C^{2,\beta}_\delta(\tilde{\Lambda}_{\alpha,\tau})$, it is necessary to have $|f|_{C^{2,\beta}_\delta} \leq \rho$ where $\rho = \mathcal{O}(\varepsilon^{1-2\delta})$ in dimension $n = 2$ or $\rho = \mathcal{O}(\varepsilon^{1-\delta})$ in higher dimensions. Therefore by the inverse function theorem of Sect. 2.5 along with Proposition 15, a solution $f := f_{\alpha,\tau}$ of the deformation problem can be found if $\tilde{\Lambda}_{\alpha,\tau}$ has been constructed so that $|\Phi_{\alpha,\tau}(0)|_{C^{0,\beta}_{\delta-2}} \leq \frac{\rho}{2C_L}$. Since Proposition 14 asserts that

$$|\Phi_{\alpha,\tau}(0)|_{C^{0,\beta}_{\delta-2}} = \mathcal{O}(\varepsilon^{(2-\delta)(3n-3)/(3n-2)}),$$

this is true so long as ε is made sufficiently small by a small enough choice of τ and δ is chosen appropriately.

As a further consequence of these estimates, the Banach space inverse function theorem asserts that the solution of the equation $\Phi_{\alpha,\tau}(f_{a,\tau}) = 0$ satisfies the estimate

$$|f_{\alpha,\tau}|_{C^{2,\beta}_\delta} = \mathcal{O}\big(C_L \varepsilon^{(2-\delta)(3n-3)/(3n-2)}\big)$$

which is much smaller that ε. Therefore the size of the perturbation of $\tilde{\Lambda}_{\alpha,\tau}$ created by the normal deformation of magnitude $f_{\alpha,\tau}$ is much smaller than the width of $\tilde{\Lambda}_{\alpha,\tau}$ at its narrowest points, i.e., in the neck regions where the width is $\mathcal{O}(\varepsilon)$. Thus $\tilde{\Lambda}_{\alpha,\tau}$ remains embedded under this normal deformation. This concludes the proof of Main Theorem 1. □

3 Gluing two Delaunay-like CMC hypersurfaces together

3.1 Introduction

The first main theorem asserts that if one positions hyperspheres of radius $\cos(\alpha)$ in a highly symmetrical manner, namely, equally spaced around an equator, then one can connect them

to each other by small catenoidal necks and perturb the resulting hypersurface into one of CMC, H_α. These hypersurfaces should be considered the analogues of the Delaunay surfaces in $\mathbb{S}^3$.

The celebrated constructions of Kapouleas [12], Mazzeo and Paccard [20] and Mazzeo et al. [21] show that it is possible to connect Delaunay surfaces in $\mathbb{R}^3$ together in a wide variety of ways and thereby deduce considerable information about the moduli space of CMCs. It is therefore tempting to try to find a similarly robust procedure in $\mathbb{S}^{n+1}$ using the gluing methods of the Sect. 2. Indeed, one can readily imagine an enormous multitude of less symmetrical configurations of hyperspheres, positioned this way and that in $\mathbb{S}^{n+1}$, only satisfying the condition that they meet each other tangentially (allowing catenoids to be inserted near the points of contact). The question becomes when the approximate solution constructed from such a configuration can be perturbed into a CMC hypersurface.

The purpose of this section is to give a construction of a CMC hypersurface based on the configuration of hyperspheres corresponding to the fusion of exactly two of the Delaunay-like hypersurfaces constructed in the previous section. The initial configuration of hyperspheres in this case will be only partially symmetric and thus the analysis that needs to be carried out will become much more complicated. One reason for this is the extra bookkeeping required to keep track of the various pieces of the initial configuration. However, a more significant reason is the presence of non-trivial approximate Jacobi fields of the linearized operator. A new technique, based on the Kapouleas [12] and Korevaar et al. [17] balancing condition, will be introduced to deal with this difficulty.

Remark The Delaunay-like CMC hypersurfaces of the previous section can be either immersed and non-compact, or immersed and compact, or embedded and compact, depending on the choices made during the construction. Such freedom is also available here, though in this section of the article only compact CMC hypersurfaces will be found.

3.2 The new approximate solution

The approximately CMC hypersurface that looks like the fusion of two Delaunay-like hypersurfaces in $\mathbb{S}^{n+1}$ will be constructed roughly as follows. Consider *two* orthogonally intersecting geodesics in $\mathbb{S}^{n+1}$. Introduce the small spacing parameter τ so that $0 < \tau \ll 2\pi$ and $2\alpha + \tau = 2\pi m/N$ for a pair of positive and relatively prime integers m and N, where N is an even integer such that $N/2$ is odd. Now position the rotated copies of S_α in a symmetric manner around both geodesics so that each rotated hypersphere is separated from its two nearest neighbours by a distance τ.

Since this initial configuration of hyperspheres is less symmetric than before, one expects to find non-trivial approximate Jacobi fields on the compact hypersurface formed by joining the hyperspheres together using normal perturbations and small catenoidal necks as before. Overcoming these Jacobi fields will require extra flexibility in the initial configuration. This will be incorporated here by introducing small *displacement parameters* for each hypersphere. These will be denoted by σ_k where $k = 0, \ldots, N-1$ and often abbreviated simply by σ. The idea is to rotate the kth hypersphere on each of the geodesics by an angle of $k(2\alpha + \tau) + \sigma_k$ which displaces it slightly from its 'natural' position. The details of the construction outlined above will now be given.

3.2.1 Positioning the hyperspheres

For $s = 1, 2$ let γ_s denote the geodesic of $\mathbb{S}^{n+1}$ formed by intersecting $\mathbb{S}^{n+1}$ with the (x^0, x^s)-plane. Define the following rotations

$$
R_{1,\theta} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & I_{n-1} \end{pmatrix} \quad \text{and} \quad R_{2,\theta} = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ \hline 0 & 0 & 0 & I_{n-1} \end{pmatrix}
$$

generating motions around γ_1 and γ_2, respectively. Here I_{n-1} is the $(n-1) \times (n-1)$ identity matrix.

Introduce the displacement parameters $\sigma_0, \ldots, \sigma_{N-1}$ and consider the set of all the hyperspheres of the form $R_{s,\sigma_k} \circ R^k_{s,2\alpha+\tau}(S_\alpha)$ for $s = 1, 2$ and $k = 0, \ldots, N-1$. This is a configuration of $2N-2$ hyperspheres wrapping m times around γ_1 and another m times around γ_2. The initial configuration of hyperspheres is then

$$
\Lambda^{\#}_{\alpha,\tau,\sigma} := \bigcup_{s=1}^{2} \bigcup_{k=0}^{N-1} R_{s,\sigma_k} \circ R^k_{s,2\alpha+\tau}(S_\alpha). \tag{30}
$$

The separation between $R_{s,\sigma_k} \circ R^k_{s,2\alpha+\tau}(S_\alpha)$ and its nearest neighbour $R_{s,\sigma_{k+1}} \circ R^{k+1}_{s,2\alpha+\tau}(S_\alpha)$ along the geodesic γ_s is $\tau_k := \tau + \sigma_{k+1} - \sigma_k$. Note that precisely two pairs of hyperspheres amongst these coincide exactly: namely, $S_\alpha = R^0_{1,2\alpha+\tau}(S_\alpha) = R^0_{2,2\alpha+\tau}(S_\alpha)$ as well as $-S_\alpha = R^{N/2}_{1,2\alpha+\tau}(S_\alpha) = R^{N/2}_{2,2\alpha+\tau}(S_\alpha)$. Note that if $m = 1$, then no other hypersurfaces overlap and the initial configuration is embedded.

It will be important to preserve as many symmetries as possible in the construction of the approximate solution. Note first that the displacement parameters along one geodesic are equal to those along the other; this is meant to ensure symmetry with respect to the rotation which takes γ_1 into γ_2. Though not strictly necessary for the success of the construction (for example, it is possible to consider a different number of windings around each geodesic), this condition does simplify the forthcoming analysis. It will be necessary, however, to impose reflection symmetries with respect to the x^0, x^1 and x^2 axes. For this to be true, the displacement parameters must satisfy

$$
\sigma_0 = \sigma_{N/2} = 0 \quad \text{and} \quad \begin{array}{c} \sigma_k = -\sigma_{N/2-k} = \sigma_{N/2+k} = -\sigma_{N-k} \\ \text{for all } k = 1, \ldots, (N-2)/4. \end{array} \tag{31}
$$

Thus it will be assumed that these conditions are in force from now on.

3.2.2 Perturbing the hyperspheres

Each hypersphere in $\Lambda^{\#}_{\alpha,\tau,\sigma}$ must be perturbed in the normal direction in order to improve its fit with the catenoids that will be used for gluing. However, there are two changes from the method used before, which need to be incorporated into the construction: the normal perturbation must be allowed to be different on each hypersphere; and the $k = 0$ and the $k = N/2$ hyperspheres must be connected to *four* neighbours.

First, choose $\varepsilon_k > 0$ and $b_k \in \mathbb{R}$ for $k = 0, \ldots, N-1$, and let $\tilde{S}_{\alpha,\varepsilon_k,b_k} := \exp(\varepsilon_k^{n-1}(G + b_k x^1)(S_\alpha)$ be the perturbed hypersphere obtained from the normal perturbation corresponding to the function $G + b_k x^1 : S_\alpha \setminus \{p^+, p^-\} \to \mathbb{R}$. Here, G is the singular solution of the equation $\mathcal{L}_\alpha(G) = 0$ that was used before, while $x^1 = x^1|_{S_\alpha}$ is the first coordinate function restricted to S_α which is also a solution of the equation $\mathcal{L}_\alpha(x^1) = 0$ on S_α. The purpose of the quantity $b_k f$ is to allow the constant term in the asymptotic expansion of $G + b_k x^1$ near $p^{\pm}$

to vary. Then for $s = 1, 2$ and all k except $k = 0, N/2$ one obtains the family of perturbed hyperspheres $R_{s,\sigma_k} \circ R_{s,2\alpha+\tau}^k(\tilde{S}_{\alpha,\varepsilon_k,b_k})$.

Next, let $p_s^{\pm} := R_{s,\alpha}^{\pm 1}(p)$ be the four points where the geodesics γ_s intersect S_α. Now let $G' : S_\alpha \setminus \{p_1^+, p_1^-, p_2^+, p_2^-\} \to \mathbb{R}$ denote the singular solution of the equation $\mathcal{L}_\alpha(G') = 0$ defined by $G' := c(G + G \circ Q)$ where c is the constant whose purpose is to give G' the same asymptotic expansion, given in Eq. 5, at all the four singular points. Similarly, let $f' = x^1|_{S_\alpha} + x^2|_{S_\alpha}$. Then for $\varepsilon_0 = \varepsilon_{N/2} > 0$ and $b_0 = b_{N/2} \in \mathbb{R}$ one obtains the two perturbed hyperspheres

$$\tilde{S}'_{\alpha,\varepsilon_0,b_0} := \exp(\varepsilon_0^{n-1}(G' + b_0 f')N_\alpha)(S_\alpha)$$
$$\tilde{S}'_{\alpha,\varepsilon_{N/2},b_{N/2}} := -\exp(\varepsilon_{N/2}^{n-1}(G' + b_{N/2}f')N_\alpha)(S_\alpha).$$

3.2.3 Inserting catenoids and assembling the approximate solution

The appropriate perturbed, truncated and re-scaled catenoidal necks must now be found so that optimal matching with the perturbed hyperspheres is achieved. However, this is a much more difficult problem than before because the non-zero displacement parameters alter the scaling parameters for each neck separately.

To begin with, choose $\bar{\varepsilon}_k > 0$ and $\bar{b}_k \in \mathbb{R}$ for $k = 0, \ldots, N-1$ and let $\tilde{\Sigma}_{\bar{\varepsilon}_k,\bar{b}_k} := \tilde{\Sigma}_{\bar{\varepsilon}_k} + \bar{b}_k$ denote the catenoidal neck defined in Sect. 2.3 but translated along its axis by an amount $\bar{b}_k$. The purpose of the parameter $\bar{b}_k$ is again to allow the constant term in the asymptotic expansion of the graphing function for the ends of the catenoid to vary. The idea is now to choose $\varepsilon_k, b_k, \bar{\varepsilon}_k, \bar{b}_k$ so that for every k, the neck $R_{s,(\sigma_k+\sigma_{k-1})/2} \circ R_{s,\alpha+\tau/2}^{2k+1} \circ K^{-1}(\tilde{\Sigma}_{\bar{\varepsilon}_k,\bar{b}_k})$ matches optimally with the perturbed hyperspheres $R_{s,\sigma_k} \circ R_{s,2\alpha+\tau}^k(\tilde{S}_{\alpha,\varepsilon_k,b_k})$ and $R_{s,\sigma_{k+1}} \circ R_{s,2\alpha+\tau}^{k+1}(\tilde{S}_{\alpha,\varepsilon_{k+1},b_{k+1}})$ in the regions of overlap. To this end, one writes the image of the kth and $(k+1)^{st}$ perturbed hyperspheres under the stereographic coordinate chart for the kth neck as graphs over the $\hat{y}$-hyperplane and equates the leading terms of the asymptotic expansions of the graphing functions given in Eq. 12 with those of the graphing function of the catenoid.

For the positive end of the neck overlapping with the $(k+1)^{st}$ perturbed hypersphere this comparison yields the equations

$$\bar{\varepsilon}_k \log(2/\bar{\varepsilon}_k) + \bar{\varepsilon}_k \log(\|\hat{y}\|) + \bar{b}_k = \tan(\tau_k/4) - \varepsilon_{k+1}\big(-c_{2k} - C_{2k}\log(\|\hat{y}\|) + b_{k+1}\big) \quad n = 2$$
$$\bar{\varepsilon}_k c_n - \frac{\bar{\varepsilon}_k^{n-1}}{(n-2)\|\hat{y}\|^{n-2}} + \bar{b}_k = \tan(\tau_k/4) - \varepsilon_{k+1}^{n-1}\left(\frac{C_{nk}}{(n-2)\|\hat{y}\|^{n-2}} + b_{k+1}\right) \quad n \geq 3$$
(32a)

where C_{nk} and c_{2k} are the quantities in (13) with τ replaced by τ_k and c_n is the constant appearing in (14). For the negative end of the neck overlapping with the kth perturbed hypersphere one finds

$$\bar{\varepsilon}_k \log(2/\bar{\varepsilon}_k) + \bar{\varepsilon}_k \log(\|\hat{y}\|) - \bar{b}_k = \tan(\tau_k/4) - \varepsilon_k\big(-c_{2k} - C_{2k}\log(\|\hat{y}\|) - b_k\big) \quad n = 2$$
$$\bar{\varepsilon}_k c_n - \frac{\bar{\varepsilon}_k^{n-1}}{(n-2)\|\hat{y}\|^{n-2}} - \bar{b}_k = \tan(\tau_k/4) - \varepsilon_k^{n-1}\left(\frac{C_{nk}}{(n-2)\|\hat{y}\|^{n-2}} - b_k\right) \quad n \geq 3$$
(32b)

To solve equations (32), it is clear that one must choose $\varepsilon_k := \varepsilon$ for all k, where $\varepsilon > 0$ is still an unknown quantity; and then one must choose $\bar{\varepsilon}_k = \varepsilon C_{2k}$ for all k in dimension $n = 2$ and

$\bar{\varepsilon}_k = \varepsilon C_{nk}^{1/(n-1)}$ for all k in dimension $n \geq 3$. The remaining equations for b_k and $\bar{b}_k$ then become

$$\bar{b}_k + \frac{\varepsilon^{n-1}}{2}(b_{k+1} + b_k) = 0$$

$$\frac{\varepsilon^{n-1}}{2}(b_{k+1} - b_k) = T_{nk} \tag{33}$$

where $T_{2k} = \tan(\tau_k/4) + \varepsilon C_{2k}(1 - \log(2/(\varepsilon C_{2k}))$ and $T_{nk} = \tan(\tau_k/4) - \varepsilon c_n C_{nk}^{1/(n-1)}$. One can check that this $2N \times 2N$ linear system has a one-dimensional kernel spanned by $b_k = \varepsilon^{1-n}$ and $\bar{b}_k = -1$ for all k and that the condition $\sum_{k=0}^{N-1} T_{nk} = 0$ guarantees that a solution of (33) exists. This condition can be rearranged as

$$\sum_{k=0}^{N-1} \tan(\tau_k/4) = \begin{cases} \varepsilon \sum_{k=0}^{N-1} C_{2k}\big(\log\big(2/(\varepsilon C_{2k})\big) - 1\big) & n = 2 \\ \varepsilon \sum_{k=0}^{N-1} c_n C_{nk}^{1/(n-1)} & n \geq 3 \end{cases} \tag{34}$$

so that the parameter ε can be determined uniquely from τ and the displacement parameters in this way. One can then find a solution, for b_k and $\bar{b}_k$ that satisfies $b_0 = 0$ and thus $b_{N/2} = 0$ by the symmetry condition (31). It can be shown that for $k = 1, \ldots, \frac{1}{2}(N/2 - 1)$ quantities on the kth, $(N/2 - k - 1)$st, $(N/2 + k)$th and $(N - k - 1)$st necks are equal and quantities on the kth, $(N/2 - k)$th, $(N/2 + k)$th and $(N - k)$th perturbed hyperspheres are equal for the same reason. This solution satisfies the estimates $\max_k\big(\varepsilon^{n-1}|b_k| + |\bar{b}_k|\big) = \mathcal{O}(\varepsilon)$.

Remark The estimate for $|b_k|$ will ensure that the errors introduced in the mean curvature of S_α by the perturbation corresponding to $\varepsilon_k^{n-1}(G + b_k x^1)$ are of the same magnitude as in Proposition 14 when the $b_k x^1$ term was absent.

Once the parameters ε_k, b_k, $\bar{\varepsilon}_k$ and $\bar{b}_k$ have been chosen according to the procedure above, one can then assemble the approximate solution in the same way as before. Note that the condition $b_0 = b_{N/2} = 0$ ensures proper matching of all four perturbed hyperspheres near $\tilde{S}'_{\alpha,\varepsilon_0}$ and $\tilde{S}'_{\alpha,\varepsilon_{N/2}}$.

Definition 16 Recall the terminology from Sect. 2.3.

- Define the neck regions

$$\mathcal{N}_\varepsilon^{s,k,\pm} := R_{s,(\sigma_k+\sigma_{k+1})/2} \circ R_{s,\alpha+\tau/2}^{2k+1} \circ K^{-1}\big(\tilde{\Sigma}_{\bar{\varepsilon}_k,\bar{b}_k}^{\pm} \cap Cyl(\rho_\varepsilon/2)\big).$$

Set $\mathcal{N}_\varepsilon^{s,k} := \mathcal{N}_\varepsilon^{s,k,+} \cup \mathcal{N}_\varepsilon^{s,k,-}$. These necks are oriented so that $\mathcal{N}_\varepsilon^{s,k,+}$ lies ahead of $\mathcal{N}_\varepsilon^{s,k,-}$ along the geodesic γ_s, and that the kth neck is centred on the point $R_{s,(\sigma_k+\sigma_{k+1})/2} \circ R_{s,\alpha+\tau/2}^{2k+1}(p)$.

- Define the transition regions

$$\mathcal{T}_\varepsilon^{s,k,\pm} := R_{s,(\sigma_k+\sigma_{k+1})/2} \circ R_{s,\alpha+\tau/2}^{2k+1} \circ K^{-1}\big(\tilde{\Sigma}_{\bar{\varepsilon}_k,\bar{b}_k}^{\pm} \cap [Cyl(2\rho_\varepsilon) \setminus Cyl(\rho_\varepsilon/2)]\big).$$

Set $\mathcal{T}_\varepsilon^{s,k} := \mathcal{T}_\varepsilon^{s,k,+} \cup \mathcal{T}_\varepsilon^{s,k,-}$.

- Define the exterior regions

$$\mathcal{E}_\varepsilon^0 := \tilde{S}'_{\alpha,\varepsilon_0,b_0} \setminus \big[B_{\tilde{\rho}_\varepsilon}(p_1^+) \cup B_{\tilde{\rho}_\varepsilon}(p_1^-) \cup B_{\tilde{\rho}_\varepsilon}(p_2^+) \cup B_{\tilde{\rho}_\varepsilon}(p_2^-)\big]$$

$$\mathcal{E}_\varepsilon^{N/2} := -\mathcal{E}_\varepsilon^0$$
$$\mathcal{E}_\varepsilon^{s,k} := R_{s,\sigma_k} \circ R_{s,2\alpha+\tau}^k \left(\tilde{S}_{\alpha,\varepsilon_k,b_k} \setminus \left[B_{\tilde{\rho}_\varepsilon}(p_s^+) \cup B_{\tilde{\rho}_\varepsilon}(p_1^-) \right] \right).$$

The approximate solution corresponding to the configuration $\Lambda_{\alpha,\tau,\sigma}^{\#}$ can now be defined as follows.

Definition 17 The *approximate solution* with parameters α, τ and σ is the hypersurface

$$\tilde{\Lambda}_{\alpha,\tau,\sigma} := \mathcal{E}_\varepsilon^0 \cup \mathcal{E}_\varepsilon^{N/2} \cup \left[\bigcup_{s=1}^{2} \bigcup_{k=1}^{N/2-1} \left(\mathcal{E}_\varepsilon^{s,k} \cup \mathcal{E}_\varepsilon^{s,N-k} \cup \mathcal{T}_\varepsilon^{s,k} \cup \mathcal{N}_\varepsilon^{s,k} \cup \mathcal{T}_\varepsilon^{s,N-k} \cup \mathcal{N}_\varepsilon^{s,N-k} \right) \right].$$
(35)

This definition contains no redundancy.

3.3 Symmetries of the approximate solution

The approximate solution $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ is considerably less symmetric than before since symmetry with respect to each $R_{s,2\alpha+\tau}$ separately no longer holds. However, it is important to realize that $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ does still possess many symmetries that can be exploited to reduce the number of approximate Jacobi fields that will have to be dealt with during the perturbation of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ into a CMC hypersurface. These symmetries are as follows.

1. *Transverse rotational symmetries.*
   Define the transformation $S_B^{012} \in O(n+2)$ by choosing $B \in O(n-1)$ and then setting

$$S_B^{012} := \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ & & 1 & \\ \hline 0 & & & B \end{pmatrix}$$

   which is the transformation keeping the x^0, x^1 and x^2 coordinates fixed while transforming the remaining coordinates by B. Each S_B^{012} is a symmetry of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ by arguments similar to those for the analogous symmetries in the construction of the Delaunay-like hypersurfaces in the previous section.

2. *Reflection symmetries.*
   For $s \in 0, 1, 2$, define the reflections $T_s \in O(n+2)$ by

$$T_s(x^0, \ldots, x^n) := (x^0, \ldots, -x^s, \ldots, x^n).$$

   These reflections satisfy: $T_s \circ R_{1,\theta} \circ T_s = R_{1,\theta}^{-1}$ for $s = 0, 1$ and $T_s \circ R_{2,\theta} \circ T_s = R_{2,\theta}^{-1}$ for $s = 0, 2$; as well as $T_1 \circ R_{2,\theta} \circ T_1 = R_{2,\theta}$ and $T_2 \circ R_{1,\theta} \circ T_2 = R_{1,\theta}$. The initial configuration of unconnected hyperspheres $\Lambda_{\alpha,\tau,\sigma}^{\#}$ is invariant under these symmetries because the displacement parameters satisfy (31). Since the neck regions are exactly determined by the separation and displacement parameters (via the neck scale parameters) further consequences are: T_0 reflects each neck region parallel to its axis and then permutes it with another neck region on the same geodesic; T_1 reflects the neck regions on γ_1 parallel to their axis and permutes them amongst themselves and reflects the neck regions on γ_2 perpendicular to their axis and permutes them amongst themselves; and T_2 reflects the neck regions on γ_2 parallel to their axis and permutes them amongst themselves and reflects the neck regions on γ_1 perpendicular to their axis and permutes them amongst themselves. Hence $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ is invariant under T_0, T_1, T_2.

3. *Parallel rotational symmetry.*

Let $Q \in SO(n+1)$ denote the rotation

$$
Q := \left(\begin{array}{ccc|c}
1 & 0 & 0 & \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & \\
\hline
& 0 & & I
\end{array} \right) \tag{36}
$$

which rotates γ_1 into γ_2. Furthermore, $Q \circ R_{1,\theta} \circ Q^{-1} = R_{2,\theta}$ so that $\Lambda^{\#}_{\alpha,\tau,\sigma}$ is invariant under Q. Also, Q rotates each neck region on γ_1 to a neck region on γ_2. Hence $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ is invariant under Q.

3.4 The analytic set-up

For each sufficiently small choice of displacement parameter σ, the approximate solution $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ has mean curvature almost equal to H_α everywhere except in the neck regions. Each of these hypersurfaces is thus a candidate for deformation into an exactly CMC hypersurface using the methods of the previous section. However, it will turn out that because of the loss of symmetry, only a specific choice of σ leads to a candidate for which the deformation can be carried out successfully, and identifying this σ requires new techniques.

The analytic set-up for the present deformation problem, however, begins in the same way as before. The approximate solution $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ will be deformed equivariantly with respect to the symmetries T_0, T_1, T_2, Q and S^{012}_B for all $B \in SO(n-2)$ by using normal deformations generated by functions invariant under these symmetries.

Definition 18 Let $X := \{ f : \tilde{\Lambda}_{\alpha,\tau,\sigma} \to \mathbb{R} : f \circ T_s = f \circ S^{012}_B = f \circ Q = f \;\; \forall s = 0, 1, 2 \text{ and } B \in SO(n-2) \}$.

Next, let $\phi_f(y) := \exp_y(f(y) \cdot N(y))$ denote the deformation of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ obtained by taking the exponential of the vector field $f \cdot N$ where N is the unit normal vector field of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ and f is a function on $\tilde{\Lambda}_{\alpha,\tau,\sigma}$.

Definition 19 Let $\Phi_{a,\tau,\sigma}$ be the operator $f \longmapsto H_{\phi_f(\tilde{\Lambda}_{\alpha,\tau,\sigma})} - H_\alpha$ with linearization at zero $\tilde{\mathcal{L}}_{\alpha,\tau,\sigma}$.

Finally, let $C^{k,\beta}_\delta$ denote a weighted Hölder space defined as in Sect. 2.6 except where the weight function is modified to take into account the additional perturbed hyperspheres and neck regions in $\tilde{\Lambda}_{\alpha,\tau,\sigma}$. The operator $\Phi_{a,\tau,\sigma} : C^{2,\beta}_\delta(X) \to C^{0,\beta}_{\delta-2}(X)$ is a well-defined, non-linear partial differential operator on $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ whose zero, if one can be found, corresponds to a deformation of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ to an exactly CMC hypersurface.

The Banach space inverse function theorem will once again be used to find a zero of $\Phi_{a,\tau,\sigma}$ in $C^{2,\beta}_\delta(X)$ for the right choice of σ. However, it turns out that not all the approximate Jacobi fields of $\tilde{\mathcal{L}}_{\alpha,\tau,\sigma}$ are excluded by the symmetries. To see which ones, proceed as follows. First define smooth and monotone cut-off functions $\chi^{s,k}$ for $k \neq 0, N/2$ as well as χ^0 and $\chi^{N/2}$ on $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ that are supported, respectively, in subsets of $\mathcal{E}^{s,k}_\varepsilon$, $\mathcal{E}^0_\varepsilon$ and $\mathcal{E}^{N/2}_\varepsilon$ that will be made precise later. Let x^t be the t-th coordinate function for $t = 1, \ldots, n$. Define the following functions on $\tilde{\Lambda}_{\alpha,\tau,\sigma}$:

$$\tilde{q}_{s,k}^t := \chi^{s,k} \cdot \left(x^t \big|_{\tilde{S}_{\alpha,\varepsilon_k,b_k}} \circ R_{s,2\alpha+\tau}^{-k} \circ R_{s,\sigma_k}^{-1} \right)$$

$$\tilde{q}_0^t := \chi^0 \cdot \left(x^t \big|_{\tilde{S}_{\alpha,\varepsilon_0}'} \right) \tag{37}$$

$$\tilde{q}_{N/2}^t := \chi^{N/2} \cdot \left(x^t \big|_{-\tilde{S}_{\alpha,\varepsilon_{N/2}}'} \right).$$

Then according to the discussion of Sect. 2.5, the vector space of *all* approximate Jacobi fields in $C^{2,\beta}(\tilde{\Lambda}_{\alpha,\tau,\sigma})$ is $\tilde{\mathcal{K}} := \mathrm{span}_{\mathbb{R}}\{\tilde{q}_{s,k}^t : \text{all } s, t, k\}$. It remains to identify the subspace of these functions which have the symmetries needed to belong to $C_\delta^{2,\beta}(X)$. By using the commutation relations between the various symmetries and the rotations $R_{s,\theta}$, one finds the following subspace.

Definition 20 The subspace of approximate Jacobi fields of $\tilde{\mathcal{L}}_{\alpha,\tau,\sigma}$ in $C_\delta^{2,\beta}(X)$ is the subspace of functions

$$\tilde{\mathcal{K}}_{sym} := \mathrm{span}_{\mathbb{R}} \left\{ \sum_{s=1}^2 \left(\tilde{q}_{s,k}^1 - \tilde{q}_{s,N/2-k}^1 + \tilde{q}_{s,N/2+k}^1 - \tilde{q}_{s,N-k}^1 \right) : k = 1, \dots, \frac{N-2}{4} \right\}.$$

Denote the functions spanning $\tilde{\mathcal{K}}_{sym}$ by $\tilde{q}_k$. Note that $\dim(\tilde{\mathcal{K}}_{sym}) = (N-2)/4$.

Essentially, each function $\tilde{q}_k$ is identical on the kth, $(N/2-k)$th, $(N/2+k)$th and $(N-k)$th perturbed hyperspheres, and these functions pull back to the coordinate function x^1 restricted to $\tilde{S}_{\alpha,\varepsilon_k,b_k}$.

3.5 The new linear estimate

Because of the presence of the approximate Jacobi fields in $\tilde{\mathcal{K}}_{sym}$, the linearized operator $\tilde{\mathcal{L}}_{\alpha,\tau,\sigma}$ on $C_\delta^{2,\beta}(X)$ no longer has an inverse whose norm is uniformly bounded with respect to the singular parameter τ. One way out of this difficulty is to project $\tilde{\mathcal{L}}_{\alpha,\tau,\sigma}$ onto a subspace of $C_{\delta-2}^{0,\beta}(X)$ which is transverse to the approximate co-kernel associated to $\tilde{\mathcal{K}}_{sym}$, and construct a bounded right inverse for the projected operator. Since $\tilde{\mathcal{L}}_{\alpha,\tau,\sigma}$ is self-adjoint, an appropriate subspace to choose is the L^2-orthogonal complement of $\tilde{\mathcal{K}}_{sym}$, denoted $X^\perp$. Let $\pi : X \to X^\perp$ denote the L^2 projection onto $X^\perp$ and set $\tilde{\mathcal{L}}_{\alpha,\tau,\sigma}^\perp := \pi \circ \tilde{\mathcal{L}}_{\alpha,\tau,\sigma}$.

Proposition 21 *Suppose first that the dimension of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ is $n \geq 3$. Choose $\delta \in (n-2, 0)$ and sufficiently small separation and displacement parameters τ and σ. Then the linearized operator $\tilde{\mathcal{L}}_{\alpha,\tau,\sigma}^\perp : C_\delta^{2,\beta}(X) \to C_{\delta-2}^{0,\beta}(X^\perp)$ possesses a bounded right inverse $\tilde{\mathcal{R}}_{\alpha,\tau,\sigma}$ satisfying the estimate*

$$|\tilde{\mathcal{R}}_{\alpha,\tau,\sigma}(f)|_{C_\delta^{2,\beta}} \leq C|f|_{C_{\delta-2}^{0,\beta}}$$

where C is a constant independent of τ and σ. If the dimension of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ is $n = 2$ then one can choose $\delta \in (-1, 0)$ and find a right inverse satisfying the estimate

$$|\tilde{\mathcal{R}}_{\alpha,\tau,\sigma}(f)|_{C_\delta^{2,\beta}} \leq C\varepsilon^\delta|f|_{C_{\delta-2}^{0,\beta}}$$

where C is a constant independent of τ and σ.

Proof The proof of this result is mostly the same as the proofs of Proposition 12 and Proposition 13, but with a few essential differences. The idea is to solve the equation $\tilde{\mathcal{L}}_{\alpha,\tau,\sigma}(u) = \pi(f)$ in $C_\delta^{2,\beta}(X)$ using the patching technique to obtain $u := \tilde{\mathcal{R}}_{\alpha,\tau,\sigma}(f)$ and then to estimate $|u|_{C_\delta^{2,\beta}}$ in terms of $|f|_{C_{\delta-2}^{0,\beta}}$.

For any $\rho \in (\tilde{\rho}_\varepsilon, \rho_0)$ define the enlarged neck regions and exterior regions $\mathcal{N}_\varepsilon^{s,k}(\rho)$ and $\mathcal{E}_\varepsilon^{s,k}(\rho)$ for each $s = 1, 2$ and $k = 1, \ldots, N/2 - 1, N/2 + 1, \ldots, N - 1$ in the same way as before. Define also the cut-off function $\chi_{neck,\rho}^{s,k}$ and $\chi_{ext,\rho}^{s,k}$ corresponding to $\mathcal{N}_\varepsilon^{s,k}(\rho)$ and $\mathcal{E}_\varepsilon^{s,k}(\rho)$, respectively, as well as the cut-off functions $\eta_{neck}^{s,k}$ and $\eta_{ext}^{s,k}$ corresponding to $\mathcal{N}_\varepsilon^{s,k}$ and $\mathcal{E}_\varepsilon^{s,k}$ respectively. Finally, define enlarged regions $\mathcal{E}_\varepsilon^0(\rho)$ and $\mathcal{E}_\varepsilon^{N/2}(\rho)$ and their corresponding cut-off functions in an analogous manner (taking into account that $\mathcal{E}_\varepsilon^0$ and $\mathcal{E}_\varepsilon^{N/2}$ are perturbed hyperspheres with four points removed). Suppose now that $n \geq 3$. Choose $\pi(f) \in C_\delta^{2,\beta}(X^\perp)$ with $\delta \in (2 - n, 0)$ and decompose $\pi(f)$ as

$$\pi(f) = f_{ext}^0 + f_{ext}^{N/2} + \sum_{s=1}^{2} \left(\sum_{k=1}^{N/2-1} \left(f_{ext}^{s,k} + f_{ext}^{s,N-k} \right) + \sum_{k=0}^{N-1} f_{neck}^{s,k} \right)$$

where

$$\begin{aligned} f_{ext}^0 &:= \chi_{ext}^0 \cdot \pi(f) \\ f_{ext}^{N/2} &:= \chi_{ext}^{N/2} \cdot \pi(f) \end{aligned} \quad \text{and} \quad \begin{aligned} f_{ext}^{s,k} &:= \chi_{ext}^{s,k} \cdot \pi(f) \\ f_{neck}^{s,k} &:= \chi_{neck}^{s,k} \cdot \pi(f). \end{aligned}$$

Remark Use $\chi_{ext}^{s,k}$, χ_{ext}^0 and $\chi_{ext}^{N/2}$ as the cut-off functions in Definition 20.

Observe that the following symmetries hold for these various functions. First, the invariance with respect to Q and T_0, T_1, T_2 implies $f_{ext}^0 = f_{ext}^{N/2}$ and

$$f_{ext}^{1,k} = f_{ext}^{1,N/2-k} = f_{ext}^{1,N/2+k} = f_{ext}^{1,N-k} = f_{ext}^{2,k} = f_{ext}^{2,N/2-k} = f_{ext}^{2,N/2+k} = f_{ext}^{2,N-k}$$

$$f_{neck}^{1,k} = f_{neck}^{1,N/2-k-1} = f_{neck}^{1,N/2+k} = f_{neck}^{1,N-k-1} = f_{neck}^{2,k} = f_{neck}^{2,N/2-k-1} = f_{neck}^{2,N/2+k} = f_{neck}^{2,N-k-1}.$$

In addition, $f_{neck}^{1,(N-2)/4}$ is an even function. Next, choose $k = 1, \ldots, N/2 - 1$ and consider what effect the invariance with respect to S_B^{012} for $B \in O(n - 1)$ and T_{3-s} has on these functions. The function $f_{neck}^{s,k}$ can be pulled back to a function on $\tilde{\Sigma}_{\bar{\varepsilon}_k}$. Write the pull back function as $f_{neck}^{s,k} := f_{neck}^{s,k}(y^2, y^3, \ldots, y^{n-1})$ using a parametrization for $\tilde{\Sigma}_{\bar{\varepsilon}_k}$ as a cylindrically symmetric graph in $\mathbb{R} \times \mathbb{R}^n$. Then one can check that $f_{neck}^{s,k}$ satisfies the symmetries

$$f_{neck}^{s,k}(y^2, B(y^3, \ldots, y^{n+1})) = f_{neck}^{s,k}(-y^2, y^3, \ldots, y^{n+1}) = f_{neck}^{s,k}(y^2, y^3, \ldots, y^{n+1}) \quad (38)$$

where B acts on $(y^3, \ldots y^{n+1}) \in \mathbb{R}^{n-1}$ in the obvious way. Similarly, the function $f_{ext}^{s,k}$ can be pulled back to a function on $\tilde{S}_{\alpha,\varepsilon_k,b_k}$. Write the pull-back function as $f_{ext}^{s,k} := f_{ext}^{s,k}(\mu, \Theta)$ using $(\mu, \Theta) \in (0, \pi) \times \mathbb{S}^{n-1}$ parametrization of $\tilde{S}_{\alpha,\varepsilon_k,b_k}$. Then one can check that $f_{ext}^{s,k}$ satisfies the symmetries

$$f_{ext}^{s,k}(\mu, \hat{S}_B(\Theta)) = f_{ext}^{s,k}(\mu, \hat{T}(\Theta)) = f_{ext}^{s,k}(\mu, \Theta) \quad (39)$$

where $\hat{S}_B$ is the transformation of $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ given by the matrix $\hat{S}_B := \begin{pmatrix} 1 & \\ & B \end{pmatrix}$, and $\hat{T}$ is the transformation of $\mathbb{S}^{n-1}$ given by the matrix $\hat{T} := \begin{pmatrix} -1 & \\ & I \end{pmatrix}$ where I is the $(n-1) \times (n-1)$ identity matrix. Finally, one can pull back f_{ext}^0 and $f_{ext}^{N/2}$ to functions $f_{ext}^* := f_{ext}^*(\mu, \Theta)$ on $\tilde{S}'_{\alpha, \varepsilon_0}$. Then one can check that these functions satisfy the symmetries

$$f_{ext}^*(-\mu, \Theta) = f_{ext}^*(\mu, \hat{T}(\Theta)) = f_{ext}^*(\mu, \hat{S}_B(\Theta)) = f(\mu, \Theta). \qquad (40)$$

This is because *both* symmetries T_1 and T_2 fix $\mathcal{E}_\varepsilon^0(\rho)$ and $\mathcal{E}_\varepsilon^{N/2}(\rho)$.

The first step in the patching technique is to find local solutions in the neck regions. That is, one solves the equation $\frac{1}{4}\mathcal{L}_{\varepsilon_k \Sigma}(u_{neck}^{s,k}) = f_{neck}^{s,k}$ in $C_{\delta,sym}^{2,\beta}(\varepsilon_k \Sigma)$ exactly as before (using the fact that the symmetries (38) ensure that $C_{\delta,sym}^{2,\beta}(\varepsilon_k \Sigma)$ for $\delta \in (2-n, 0)$ contains none of the Jacobi fields of $\mathcal{L}_{\varepsilon_k \Sigma}$ listed in (20) and thus $\frac{1}{4}\mathcal{L}_{\varepsilon_k \Sigma}$ is bijective with δ in this range). The solution can then be extended to all of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ as before and the extended function $\bar{u}_{neck}^{s,k}$ satisfies the estimate $|\bar{u}_{neck}^{s,k}|_{C_\delta^{2,\beta}} \leq C|\pi(f)|_{C_{\delta-2}^{0,\beta}}$.

The next step is to find local solutions on the exterior regions. Choose a small $\kappa \in (0, 1)$ and define the functions

$$\hat{f}_{ext}^{s,k} := \chi_{ext,\kappa\rho}^{s,k}\Big(\pi(f) - \sum_{s',k'} \tilde{\mathcal{L}}_{\alpha,\tau,\sigma}^\perp(\bar{u}_{neck}^{s',k'})\Big)$$

$$\hat{f}_{ext}^0 := \chi_{ext,\kappa\rho}^0\Big(\pi(f) - \sum_{s',k'} \tilde{\mathcal{L}}_{\alpha,\tau,\sigma}^\perp(\bar{u}_{neck}^{s',k'})\Big) \qquad (41)$$

$$\hat{f}_{ext}^{N/2} := \chi_{ext,\kappa\rho}^{N/2}\Big(\pi(f) - \sum_{s',k'} \tilde{\mathcal{L}}_{\alpha,\tau,\sigma}^\perp(\bar{u}_{neck}^{s',k'})\Big).$$

One can view each of the functions (41) as $C^{0,\beta}$ functions on S_α and attempt to solve the equation $\mathcal{L}_\alpha(u_{ext}^*) = \hat{f}_{neck}^*$ up to projection onto the approximate co-kernel.

Begin with $\hat{f}_{neck}^{s,k}$. The symmetries (39) imply that the only invariant Jacobi field of $\mathcal{L}_\alpha$ is the coordinate function x^1. Compute $\lambda^{s,k} := \int_{S_\alpha} \hat{f}_{ext}^{s,k} \cdot x^1 / \int_{S_\alpha} \tilde{q}_k \cdot x^1$ so that $\big(\hat{f}_{ext}^{s,k}\big)^\perp := \hat{f}_{ext}^{s,k} - \lambda^{s,k}\tilde{q}_k$ is orthogonal to x^1 where $\tilde{q}_k \in \tilde{\mathcal{K}}_{sym}$ is defined in Definition (20). The equation $\mathcal{L}_\alpha(u_{ext}^{s,k}) = \big(\hat{f}_{ext}^{s,k}\big)^\perp$ can now be solved for $u_{ext}^{s,k}$ in $C^{2,\beta}(S_\alpha)$. Moreover, it is true that

$$\int_{S_\alpha} \hat{f}_{ext}^{s,k} \cdot x^1 = \int_{S_\alpha \cap Tub_{2\rho}(\gamma)} \hat{f}_{ext}^{s,k} \cdot x^1 + \int_{S_\alpha \setminus Tub_{2\rho}(\gamma)} \hat{f}_{ext}^{s,k} \cdot x^1$$

and

$$\int_{S_\alpha \setminus Tub_{2\rho}(\gamma)} \hat{f}_{ext}^{s,k} \cdot x^1 = \int_{S_\alpha \setminus Tub_{2\rho}(\gamma)} f_{ext}^{s,k} \cdot x^1 = \int_{\tilde{\Lambda}_{\alpha,\tau,\sigma}} f \cdot \tilde{q}_k = 0$$

using the fact that $\pi(f) \in C_{\delta-2}^{0,\beta}(X^\perp)$. Thus with the help of estimates similar to (25) one deduces

$$|\lambda^{s,k}| \leq C\kappa\big(\kappa^2 + \varepsilon^{2(3n-3)/(3n-2)}\big)|\pi(f)|_{C_{\delta-2}^{0,\beta}}$$

where C_κ depends on κ and δ but not ε. The solution therefore satisfies the estimate $|u_{ext}^{s,k}|_{C^{2,\beta}(S_\alpha)} \le C_\kappa |\pi(f)|_{C_{\delta-2}^{0,\beta}}$. The remaining functions f_{ext}^0 and $f_{ext}^{N/2}$ are easier. The symmetries (40) imply that there are no invariant Jacobi fields of $\mathcal{L}_\alpha$ to worry about. Hence solutions of $\mathcal{L}_\alpha(u_{ext}^*) = f_{ext}^*$ can be found without further ado. The same kind of estimate holds for these solutions.

The extension of the various local solutions on the perturbed hyperspheres to all of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ is slightly more involved than in the previous proof. Suppose that $u_{ext}^{s,k}(p^\pm) := a_\pm^{s,k}$. If $c_0^{s,k}$ and $c_1^{s,k}$ satisfy the equations $a_+^{s,k} = c_0^{s,k} - c_1^{s,k}$ and $a^{s,k+1} = c_0^{s,k} + c_1^{s,k}$ then the leading term in the asymptotic expansion of the function $c_0^{s,k} J_0 + c_1^{s,k} J_1$ on the kth neck matches the leading term of the Taylor expansion of $u_{ext}^{s,k}$ on the overlap with the kth perturbed hypersphere and with the leading term of the Taylor expansion of $u_{ext}^{s,k+1}$ on the overlap with the $(k+1)^{st}$ perturbed hypersphere. This can be done for $k = 2, \ldots, N/2 - 1$ and $k = N/2 + 2, \ldots, N - 1$. By symmetry, the quantities $u_{ext}^{s,1}(p^-)$, $u_{ext}^{s,N/2-1}(p^+)$, $u_{ext}^{s,N/2+1}(p^-)$ and $u_{ext}^{s,N-1}(p^+)$ are all equal, as are $u_{ext}^0(p_s^\pm)$ and $u_{ext}^{N/2}(p_s^\pm)$. Hence one can add the same multiples of J_0 and J_1 in the necks corresponding to $k = 1, N/2, N/2 + 1$ and N to match with the leading term in the Taylor series of u_{ext}^0 and $u_{ext}^{N/2}$ on the regions of overlap with these necks. Finally, one arrives at the extended solution

$$\bar{u}_{ext} := \eta_{ext}^0 u_{ext}^0 + \eta_{ext}^{N/2} u_{ext}^{N/2} + \sum_{s=1}^{2} \sum_{\substack{k=1 \\ k \neq N/2}}^{N-1} \eta_{ext}^{s,k} u_{ext}^{s,k} + \sum_{s=1}^{2} \sum_{k=1}^{N} \eta_{neck}^{s,k} \left(c_0^{s,k} J_0 + c_1^{s,k} J_1 \right).$$

The extended function $\bar{u}_{ext}$ satisfies the estimate $|\bar{u}_{ext}|_{C_\delta^{2,\beta}} \le C |\pi(f)|_{C_{\delta-2}^{0,\beta}}$.

The proof now concludes as follows. Set $\bar{u} := \bar{u}_{ext} + \sum_{s,k} \bar{u}_{neck}^{s,k}$. Then estimates similar to those used before lead to the bounds

$$|\tilde{\mathcal{L}}_{\alpha,\tau,\sigma}(\bar{u}) - \pi(f)|_{C_{\delta-2}^{0,\beta}} \le C \left(\kappa^2 + \varepsilon^{2(3n-3)/(3n-2)} \right) |\pi(f)|_{C_{\delta-2}^{0,\beta}} \tag{42}$$

and $|\bar{u}|_{C_\delta^{2,\beta}} \le C |\pi(f)|_{C_{\delta-2}^{0,\beta}}$ for constants C independent of ε. Since the factor in (42) that is in front of $|\pi(f)|_{C_{\delta-2}^{0,\beta}}$ can be made as small as desired by choosing κ and ε sufficiently small, an exact solution of $\tilde{\mathcal{L}}_{\alpha,\tau,\sigma}^\perp(u) = f$ can be found by iteration. It satisfies the bound $|u|_{C_\delta^{2,\beta}} \le C |\pi(f)|_{C_{\delta-2}^{0,\beta}}$. This completes the proof in the case $n \ge 3$. The case $n = 2$ is similar, except that the double indicial root of $\tilde{\mathcal{L}}_{\alpha,\tau,\sigma}$ must be dealt with by matching asymptotic expansions as in Proposition 13 and one ends up with the estimate $|u|_{C_\delta^{2,\beta}} \le C \varepsilon^\delta |\pi(f)|_{C_{\delta-2}^{0,\beta}}$. $\square$

3.6 The solution of the non-linear problem up to finite-dimensional error

The new linear estimate derived in the previous section and non-linear estimates identical to those of Sect. 2.8 can now be combined to solve the equation $\Phi_{a,\tau,\sigma}(f) = 0$ using the Banach space inverse function theorem up to a finite-dimensional error term in the following sense. Let π denote once again the L^2-projection onto $X^\perp$.

Proposition 22 *If τ and $\|\sigma\|$ are sufficiently small, then there exists $f_{\alpha,\tau,\sigma} \in C_\delta^{2,\beta}(X)$ satisfying $\pi \circ \Phi_{\alpha,\tau,\sigma}(f_{\alpha,\tau,\sigma}) = 0$ and there exists a constant C independent of τ and σ so that*

$$|f_{\alpha,\tau,\sigma}|_{C_\delta^{2,\beta}} \le C \cdot C_L(\varepsilon) \varepsilon^{(2-\delta)(3n-3)/(3n-2)}$$

where $C_L = \mathcal{O}(1)$ in dimension $n \geq 3$ and $C_L = \mathcal{O}(\varepsilon^\delta)$ in dimension $n = 2$. Therefore the hypersurface obtained by deforming $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ in the normal direction by an amount determined by $f_{\alpha,\tau,\sigma}$ has CMC and is embedded if $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ is.

Proof The linearization of $\pi \circ \Phi_{a,\tau,\sigma}$ at zero is $\mathrm{D}(\pi \circ \Phi_{a,\tau,\sigma})(0) = \tilde{\mathcal{L}}^\perp_{\alpha,\tau,\sigma}$ and this operator possesses a bounded right inverse by Proposition 21. The Banach space inverse function theorem can thus be applied to the equation $\pi \circ \Phi_{a,\tau,\sigma}(f) = 0$ provided that the three fundamental estimates (17), (18) and (19) described in Sect. 2.5 can be established. The construction of the right inverse and its bound in Proposition 21 constitutes the first of these estimates. The second and third estimates can be proved by very straightforward modifications of the analogous estimates for Main Theorem 1, namely, Proposition 14 and Proposition 15. The proof then follows as before. □

3.7 The balancing problem and the conclusion of the proof

3.7.1 The balancing map

Proposition 22 shows that the equation $\Phi_{a,\tau,\sigma}(f) = 0$ can be solved up to a finite dimensional error term; i.e., a function $f_{\alpha,\tau,\sigma} \in C^{2,\beta}_\delta(X^\perp)$ can be found so that only the L^2-projection of $\Phi_{a,\tau,\sigma}(f_{\alpha,\tau,\sigma})$ to the subspace $\tilde{\mathcal{K}}_{sym}$ fails to vanish identically as a function of σ. It will now be shown, however, that there is a special choice of σ that yields a so-called *balanced* initial configuration $\Lambda^\#_{\alpha,\tau,\sigma}$ for which $\Phi_{a,\tau,\sigma}(f_{\alpha,\tau,\sigma})$ vanishes completely. Therefore the solution $f_{\alpha,\tau,\sigma}$ for this choice of σ yields the desired deformation of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ into an exactly CMC hypersurface.

In order to derive the *balancing condition* that will be used to determine this special value of σ, one must understand in greater detail the relationship between σ and the quantity $(\mathrm{id} - \pi) \circ \Phi_{a,\tau,\sigma}(f_{\alpha,\tau,\sigma})$ where π is the L^2-projection onto $X^\perp$. To this end, introduce the functions $\tilde{q}^*_k : \tilde{\Lambda}_{\alpha,\tau,\sigma} \to \mathbb{R}$ which are obtained from the functions in Definition 20 by replacing the cut-off functions $\chi^{s,k}_{ext}$ by the cut-off functions $\eta^{s,k}_{ext}$ used in the proof of Proposition 21.

Definition 23 The *balancing map* of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ is the function $B_{\alpha,\tau} : \mathbb{R}^{(N-2)/4} \to \mathbb{R}^{(N-2)/4}$ given by

$$B_{\alpha,\tau}(\sigma) := \left(\int_{\tilde{\Lambda}_{\alpha,\tau,\sigma}} \Phi_{a,\tau,\sigma}(f_{\alpha,\tau,\sigma}) \cdot \tilde{q}^*_1, \ldots, \int_{\tilde{\Lambda}_{\alpha,\tau,\sigma}} \Phi_{a,\tau,\sigma}(f_{\alpha,\tau,\sigma}) \cdot \tilde{q}^*_{(N-2)/4} \right), \quad (43)$$

where $f_{\alpha,\tau,\sigma} \in C^{2,\beta}_\delta(X^\perp)$ is the solution found in Proposition 22. Note that the dimension of the domain of $B_{\alpha,\tau}$ is accurate because of the conditions (31) that are assumed to be valid for σ.

In terms of the balancing map, what remains to be done in order to prove Main Theorem 2 is to find a value of σ for which $B_{\alpha,\tau}(\sigma) = 0$. This is because the identity $\int_{\tilde{\Lambda}_{\alpha,\tau,\sigma}} \tilde{q}_j \cdot \tilde{q}^*_k = \delta_{jk} + \mathcal{O}(\rho^n)$ implies that $(\mathrm{id} - \pi) \circ \Phi_{a,\tau,\sigma}(f_{\alpha,\tau,\sigma}) = 0$ as well.

The ordinary inverse function theorem for smooth functions on Euclidean space will be used to find this value of σ in the following way. First, $B_{\alpha,\tau}$ will be approximated by a simpler function $\mathring{B}_{\alpha,\tau} : \mathbb{R}^{(N-2)/4} \to \mathbb{R}^{(N-2)/4}$ depending only on the geometry of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ (i.e., independent of $f_{\alpha,\tau,\sigma}$) using the ideas behind the Korevaar et al. [17] and Kapouleas

[12] balancing formula. There will be a fairly obvious choice of σ for which $\mathring{B}_{\alpha,\tau}(\sigma) = 0$; consequently, it will be true that $B_{\alpha,\tau}(\sigma) \approx 0$ for this choice of σ. Then by the inverse function theorem, it will be possible to find a small perturbation of σ for which $B_{\alpha,\tau}(\sigma) = 0$ exactly since it will turn out that the derivative $DB_{\alpha,\tau}(\sigma)$ will be invertible with large norm.

3.7.2 The proof of Main Theorem 2

The first task is to approximate the balancing map of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ by an easily understood function that is independent of $f_{\alpha,\tau,\sigma}$. Let $\bar{\varepsilon}_k$ denote the scale parameter of the kth neck region of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$.

Proposition 24 *The balancing map of $\tilde{\Lambda}_{\alpha,\tau}$ can be approximated as follows. Whenever τ is sufficiently small, There exists a function $\mathring{B}_{\alpha,\tau} : \mathbb{R}^{(N-2)/4} \times \mathbb{R}^{(N-2)/4}$ satisfying the estimate*

$$\|\tilde{B}_{\alpha,\tau}(\sigma)\|_{C^2} \leq C\rho_\varepsilon^n |f_{\alpha,\tau,\sigma}|_{C_\delta^{2,\beta}}$$

for some constant C independent of ε so that

$$B_{\alpha,\tau}(\sigma) = \mathring{B}_{\alpha,\tau}(\sigma) + \tilde{B}_{\alpha,\tau}(\sigma)$$

where

$$\mathring{B}_{\alpha,\tau}(\sigma) = \omega \begin{pmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \\ & & & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \bar{\varepsilon}_0^{n-1} \\ \vdots \\ \bar{\varepsilon}_{(N-2)/4}^{n-1} \end{pmatrix}. \tag{44}$$

Here ω is another constant independent of ε.

Proof Let $[B_{\alpha,\tau}(\sigma)]_k$ denote the kth component of the balancing map. By symmetry, it is sufficient for the analysis of $[B_{\alpha,\tau}(\sigma)]_k$ to examine only the kth spherical region along the geodesic γ_1 and its two adjoining neck and transition regions, namely, the region

$$\mathcal{S}_\varepsilon^k := \mathcal{T}_\varepsilon^{1,k,+} \cup \mathcal{N}_\varepsilon^{1,k,+} \cup \mathcal{E}_\varepsilon^k \cup \mathcal{T}_\varepsilon^{1,k+1,-} \cup \mathcal{N}_\varepsilon^{1,k+1,-}.$$

Next, let

$$D_k := R_{1,(\sigma_k+\sigma_{k+1})/2} \circ R_{1,\alpha+\tau/2}^{2k+1} \circ K^{-1}\left(\{(\bar{b}_k, \hat{y}) : \|\hat{y}\| \leq \varepsilon_k\}\right)$$

be the central cross-sectional n-sphere of $\mathcal{N}_\varepsilon^{1,k}$ and let $c_k := \partial D_k$ so that $\partial \mathcal{S}_\varepsilon^k = c_k \cup c_{k+1}$. Let $\phi_{\alpha,\tau,\sigma}$ denote the normal deformation of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ obtained from the solution $f_{\alpha,\tau,\sigma}$. Using symmetry and simple estimates, one finds

$$[B_{\alpha,\tau}(\sigma)]_k = 2 \int_{\phi_{a,\tau,\sigma}(\tilde{\Lambda}_{\alpha,\tau,\sigma})} \bar{\eta}_{ext}^{1,k} \cdot \left(H(\phi_{a,\tau,\sigma}(\tilde{\Lambda}_{\alpha,\tau,\sigma})) - H_\alpha\right) \cdot \left(x^1 \circ R_{1,k(2\alpha+\tau)+\sigma_k}^{-1}\right)$$

$$= 2 \int_{\phi_{a,\tau,\sigma}(\mathcal{S}_\varepsilon^{1,k})} \left(H(\phi_{a,\tau,\sigma}(\mathcal{S}_\varepsilon^{1,k})) - H_\alpha\right) \cdot \left(x^1 \circ R_{1,k(2\alpha+\tau)+\sigma_k}^{-1}\right) + \mathcal{O}(\rho_\varepsilon^n |f_{\alpha,\tau,\sigma}|_{C_\delta^{2,\beta}})$$

since $\eta_{ext}^{1,k}$ is supported outside the neighbourhood $Tub_{\rho_\varepsilon}(\gamma_1)$. Here, $\mathcal{O}(\rho_\varepsilon^n |f_{\alpha,\tau,\sigma}|_{C_\delta^{2,\beta}})$ denotes a function bounded in C^2 norm by $\rho_\varepsilon^n |f_{\alpha,\tau,\sigma}|_{C_\delta^{2,\beta}}$. Now recall the Korevaar et al. identity

given in Eq. 1 satisfied by the integral of the mean curvature of hypersurface against a Jacobi field. That is, if ν_k denotes the outward unit normal vector field of c_k parallel to $N_\varepsilon^{1,k}$ and V_k denotes the compatibly oriented unit normal vector field of D_k while Y_k denotes the vector field

$$Y_k := \left(R_{1,(\sigma_k+\sigma_{k+1})/2} \circ R_{\alpha+\tau/2}^{2k+1}\right)_* \circ \left(x^0 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^0}\right) \circ R_{\alpha+\tau/2}^{-2k-1} \circ R_{1,(\sigma_k+\sigma_{k+1})/2}^{-1},$$

then one finds

$$\frac{1}{2}[B_{\alpha,\tau}(\sigma)]_k = \left(\int_{\phi_{a,\tau,\sigma}(c_k)} \langle \nu_k, Y_k \rangle - H_\alpha \int_{\phi_{a,\tau,\sigma}(D_k)} \langle V_k, Y_k \rangle\right)$$

$$- \left(\int_{\phi_{a,\tau,\sigma}(c_{k+1})} \langle \nu_{k+1}, Y_k \rangle - H_\alpha \int_{\phi_{a,\tau,\sigma}(D_{k+1})} \langle V_{k+1}, Y_k \rangle\right) + \mathcal{O}\left(\rho_\varepsilon^n |f_{\alpha,\tau,\sigma}|_{C_\delta^{2,\beta}}\right)$$

$$= \int_{c_k} \langle \nu_k, Y_k \rangle - \int_{c_{k+1}} \langle \nu_{k+1}, Y_k \rangle + \mathcal{O}\left(\rho_\varepsilon^n |f_{\alpha,\tau,\sigma}|_{C_\delta^{2,\beta}}\right). \tag{45}$$

This is because the integrals over $\phi_{a,\tau,\sigma}(c_*)$, can be replaced by integrals over c_* and the integrals over $\phi_{a,\tau,\sigma}(D_*)$ can be dropped altogether at the expense of further error terms whose size is strictly smaller than $\mathcal{O}\left(\rho_\varepsilon^n |f_{\alpha,\tau,\sigma}|_{C_\delta^{2,\beta}}\right)$. Finally, the calculation of the integrals $\int_{c_*} \langle \nu_*, Y_k \rangle$ in (45) can be carried out in the stereographic coordinate chart used to define the neck regions, in which case Y_k simply becomes the translation vector field perpendicular to the axis of the neck. This calculation is very straightforward, and yields a quantity proportional to the $(n-1)$-dimensional area of c_k, thus proportional to $\bar{\varepsilon}_k^{n-1}$.   □

Recall that $\bar{\varepsilon}_k$ depends on the separation between the hyperspheres through the procedure culminating in formula (34). Thus one can see that there are universal functions $E : \mathbb{R} \to \mathbb{R}$ and $C : \mathbb{R} \to \mathbb{R}$ so that

$$\bar{\varepsilon}_k(\sigma) := E\big(C(\tau + \sigma_1 - \sigma_0) + \cdots + C(\tau + \sigma_0 - \sigma_{N-1})\big) \cdot C(\tau + \sigma_{k+1} - \sigma_k).$$

It is thus clear how to choose σ so that $\mathring{B}_{\alpha,\tau}$ vanishes: one chooses $\sigma_1 = \cdots = \sigma_{N-1} = 0$ which corresponds to an equal spacing of the hyperspheres making up $\tilde{\Lambda}_{\alpha,\tau,\sigma}$. By Proposition 24, one has $B_{\alpha,\tau}(0) = \mathcal{O}\left(\rho_\varepsilon^n |f_{\alpha,\tau,\sigma}|_{C_\delta^{2,\beta}}\right)$. Note that when this choice of σ is made, then all the various quantities on the different constituents of $\tilde{\Lambda}_{\alpha,\tau,0}$ are equal and determined uniquely by the choice of τ through the scale parameter $\varepsilon := \varepsilon(\tau)$.

It remains to show that the derivative $DB_{\alpha,\tau}(0)$ is invertible and is bounded below by a quantity much larger than $\mathcal{O}\left(\rho_\varepsilon^n |f_{\alpha,\tau,\sigma}|_{C_\delta^{2,\beta}}\right)$. Since $DB_{\alpha,\tau}(0) = D\mathring{B}_{\alpha,\tau}(0) + \mathcal{O}\left(\rho_\varepsilon^n |f_{\alpha,\tau,\sigma}|_{C_\delta^{2,\beta}}\right)$, it is sufficient to establish this fact for $D\mathring{B}_{\alpha,\tau}(0)$.

Proposition 25 *The derivative matrix of* $\mathring{B}_{\alpha,\tau}$ *at* $\sigma = 0$ *is*

$$
\mathrm{D}\mathring{B}_{\alpha,\tau}(0) = \omega(\varepsilon)
\begin{pmatrix}
0 & 1 & & & & \\
& 0 & 1 & & & \\
& & 0 & 1 & & \\
& & & \ddots & & \\
& & & & 0 & 1 \\
1 & & & & & 0
\end{pmatrix}
$$

where $\omega(\varepsilon) = \mathcal{O}(\varepsilon^{n-1})$. *This is an invertible linear transformation.*

Proof The matrix appearing in $\mathrm{D}\mathring{B}_{\alpha,\tau}(0)$ is easily found by applying the chain rule to $\mathring{B}_{\alpha,\tau}$ and the functions $\bar{\varepsilon}_k^{n-1} := [\varepsilon_k(\sigma)]^{n-1}$ at $\sigma = 0$.   □

The consequence of Proposition 25 is that the derivative $\mathrm{D}B_{\alpha,\tau}(0)$ is invertible and much larger than $B_{\alpha,\tau}(0)$. Consequently the inverse function theorem can be applied to find σ near zero for which $B_{\alpha,\tau}(\sigma) = 0$. The proof of Main Theorem 2 is now complete.   □

4 Attaching Delaunay-like handles to Clifford tori

4.1 Introduction

The techniques that have been developed so far allow the analogue of one more 'classical' problem from the theory of CMC hypersurfaces in Euclidean space to be realized in $\mathbb{S}^{n+1}$. That is, one can attempt to attach a number of Delaunay-like handles to a given CMC hypersurface and hope to perturb the resulting hypersurface to have CMC. It should be possible to pose conditions, in very general terms, for when such a construction could be realized. However it will be simplest to focus on one canonical example of the construction since this is easier for the reader to visualize and it is in keeping with the spirit of this article. Moreover, the generalization to more complicated settings will be readily believable once a simple example is shown to work.

The case that will be considered here is the attachment of a truncated Delaunay-like hypersurface in $\mathbb{S}^{n+1}$ to the generalized Clifford torus $\mathbb{T}_\alpha^{n_1,n_2}$ of type $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ where $n_1 + n_2 = n$. From what has been learned about the problem of gluing two Delaunay-like hypersurfaces together, it is intuitively clear that one must find a geodesic γ that makes contact orthogonally with $\mathbb{T}_\alpha^{n_1,n_2}$ in two places, and that hyperspheres of the same mean curvature as $\mathbb{T}_\alpha^{n_1,n_2}$ must be positioned with equal spacing along γ. The hyperspheres will then be in balanced position. What is not clear is that this can be done with enough symmetry to rule out all the Jacobi fields of $\mathbb{T}_\alpha^{n_1,n_2}$. However, by analyzing these Jacobi fields carefully, it will be found that it is indeed possible rule out these Jacobi fields and thereby realize the handle-attachment procedure.

4.2 The approximate handle-attachment

4.2.1 The generalized Clifford torus in $\mathbb{S}^{n+1}$

The generalized Clifford tori of type $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ with $n_1 + n_2 = n$ in $\mathbb{S}^{n+1}$ are defined by

$$
\mathbb{T}_\alpha^{n_1,n_2} := \left\{ (x_1, x_2) \in \mathbb{R}^{n_1+1} \times \mathbb{R}^{n_2+1} : \|x_1\| = \cos(\alpha) \text{ and } \|x_2\| = \sin(\alpha) \right\}
$$

where $\alpha \in (0, \pi/2)$. Then $\mathbb{T}_\alpha^{n_1, n_2}$ can be parametrized by $(\Theta^1, \Theta^2) \longmapsto (\cos(\alpha)\Theta^1, \sin(\alpha)\Theta^2)$ where the map $\Theta^j : \mathbb{S}^{n_j} \to \mathbb{R}^{n_j+1}$ is an embedding of $\mathbb{S}^{n_j}$ as the unit sphere in $\mathbb{R}^{n_j+1}$. The calculations of [6] give the relevant facts about the geometry of the Clifford tori. In particular, the mean curvature of $\mathbb{T}_\alpha^{n_1, n_2}$ is $H_\alpha^{Cliff} := n_2 \cot(\alpha) - n_1 \tan(\alpha)$ and the linearized mean curvature operator is given by

$$\mathcal{L}_{n_1, n_2, \alpha} := \cos^{-2}(\alpha)(\Delta_1 + n_1) + \sin^{-2}(\alpha)(\Delta_2 + n_2)$$

where Δ_1 and Δ_2 are the Laplacians of $\mathbb{S}^{n_1}$ and $\mathbb{S}^{n_2}$, respectively. One deduces that the Jacobi fields of $\mathbb{T}_\alpha^{n_1, n_2}$ are the products of the coordinate functions—i.e., $\mathcal{L}_{n_1, n_2, \alpha}(x_1^{t_1} x_2^{t_2}|_{\mathbb{T}_\alpha^{n_1, n_2}}) = 0$ for all $t_j = 0, \ldots, n_j$.

4.2.2 Assembly of the approximate solution

To begin assembling the approximate solution for the handle-attachment process, choose a canonical point $p \in \mathbb{T}_\alpha^{n_1, n_2}$ defined by $p := (\cos(\alpha)P_1, \sin(\alpha)P_2)$ where $P_j \in \mathbb{R}^{n_j+1}$ is the point $P_j := (1, 0, \ldots, 0)$. For any $\theta \in [0, 2\pi]$, define the rotation

$$R_\theta := \left(\begin{array}{c|c|c|c} \cos(\theta)\,|\,0 \cdots 0 & -\sin(\theta)\,|\,0 \cdots 0 \\ \hline 0 & & 0 & \\ \vdots & I_{n_1} & \vdots & 0 \\ 0 & & 0 & \\ \hline \sin(\theta)\,|\,0 \cdots 0 & \cos(\theta)\,|\,0 \cdots 0 \\ \hline 0 & & 0 & \\ \vdots & 0 & \vdots & I_{n_2} \\ 0 & & 0 & \end{array} \right). \tag{46}$$

The normal vector of $\mathbb{T}_\alpha^{n_1, n_2}$ at p is $N_p := -\sin(\alpha)P_1 + \cos(\alpha)P_2$. Let γ_p be the normal geodesic emanating from $\mathbb{T}_\alpha^{n_1, n_2}$ at the point p; then γ_p can be parametrized by $t \longmapsto R_t(p)$ for γ_p. Define the point $p' := (-\cos(\alpha)p_1, \sin(\alpha)p_2)$. Then

$$\gamma_p \cap \mathbb{T}_\alpha^{n_1, n_2} = \{p, p', -p, -p'\}$$

and these points occur at the parameter values $t = 0$, $t = \pi - 2\alpha$, $t = \pi$ and $t = 2\pi - 2\alpha$, respectively.

Recall that the mean curvature of a hypersphere $S_{\hat{\alpha}}$ with parameter $\hat{\alpha} \in (0, \pi/2)$ is $H_{\hat{\alpha}}^{Sph} := n \cot(\hat{\alpha})$. Therefore the hypersphere with the same mean curvature (up to sign) as $\mathbb{T}_\alpha^{n_1, n_2}$ has $\hat{\alpha} := \operatorname{arccot}\left|\frac{n_2}{n}\cot(\alpha) - \frac{n_1}{n}\tan(\alpha)\right|$. Denote the common value of the mean curvature simply by H_α. Since the length of the geodesic segment from p to $-p'$ is exactly 2σ, it is thus possible to position N rotated hyperspheres of parameter $\hat{\alpha}$ along γ_p from p to $-p'$, where each hypersphere is separated from its two nearest neighbours by a distance τ, when $2N\hat{\alpha} + (N+1)\tau = 2\alpha + 2m\pi$ for some integer $m \geq 0$. These spheres wind m times around γ_p before ending up a distance of exactly τ from $-p'$.

Remark It is clear that for every fixed $\alpha \in (0, \pi/2)$ and for every $\tau_0 > 0$ there is always *some* value of τ satisfying $0 < \tau < \tau_0$ and *some* integers N and m so that the equation $2N\hat{\alpha} + (N+1)\tau = 2\alpha + 2m\pi$ holds. It is not clear if this equation can be made to hold in this way for fixed m as well (e.g., for $m = 0$ which would correspond to embedded configurations). Since $\lim_{\alpha \to \pi/2} \frac{\alpha}{\hat{\alpha}(\alpha)} = \infty$ one can find N and α_0 so that $N\hat{\alpha}(\alpha_0) = \alpha_0$. Then since the derivative $\frac{d\hat{\alpha}}{d\alpha}$ is bounded away from zero, one can determine $\alpha := \alpha(\tau)$ near α_0 so that

$2N\hat{\alpha}(\alpha(\tau)) + (N+1)\tau = 2\alpha(\tau)$ for all sufficiently small τ. Moreover, if $|\alpha_0 - \pi/2| > T$ then one can arrange to have $|\alpha(\tau) - \pi/2| > T/2$. This ensures that the mean curvature of the configuration, though large since α_0 is near $\pi/2$, never becomes unbounded.

These considerations lead to the definition of the initial configuration of hyperspheres and Clifford torus that will be glued together below. For any $\theta \in [0, 2\pi]$ first introduce the matrix $Q_\alpha \in SO(n+2)$ by

$$Q_\alpha := \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \cdots 0 & 0 \cdots 0 \\ 0 & 0 & & \\ \vdots & \vdots & I_1 & 0 \\ 0 & 0 & & \\ \hline \sin(\alpha) & \cos(\alpha) & 0 \cdots 0 & 0 \cdots 0 \\ 0 & 0 & & \\ \vdots & \vdots & 0 & I_2 \\ 0 & 0 & & \end{pmatrix} \tag{47}$$

where I_j is the $n_j \times n_j$ identity matrix. The matrix Q_α takes the point $p_0 := (1, 0, \ldots, 0) \in \mathbb{R}^{n+2}$ to the point p, the vector $N_0 := (0, 1, 0, \ldots, 0) \in \mathbb{R}^{n+2}$ to the vector N_p, and the subspace $\{0, 0\} \times \mathbb{R}^n$ to the tangent space of $\mathbb{T}_\alpha^{n_1, n_2}$ at p. Note that $R_\theta \circ Q_\alpha = Q_{\theta+\alpha}$.

The rotation $R_\theta \circ Q_\alpha$ takes the hypersphere $S_{\hat{\alpha}}$ to another hypersphere centred on the point $R_\theta(p)$ along the geodesic γ_p. Let N, m, α, τ and $\hat{\alpha}$ satisfy $2N\hat{\alpha} + (N+1)\tau = 2\alpha + 2m\pi$ and introduce displacement parameters $\sigma_1, \ldots, \sigma_N$ as before. Define the initial configuration

$$\Lambda_{\alpha,\tau,\sigma}^\# := \left[\bigcup_{k=0}^{N-1} R_{\sigma_k}^{-1} \circ R_{k(2\alpha+\tau)+\alpha+\tau}^{-1} \circ Q_\alpha(S_{\hat{\alpha}}) \right] \cup \mathbb{T}_\alpha^{n_1,n_2}. \tag{48}$$

The mean curvature of each component of $\Lambda_{\alpha,\tau,\sigma}^\#$ is H_α.

The next step in the assembly of the approximate solution is the perturbation of each of the components of $\Lambda_{\alpha,\tau,\sigma}^\#$ in order to improve their fit with the catenoidal necks that are needed to glue everything together. By analogy with the procedure described in Sect. 3.2, each hypersphere $R_{k(2\alpha+\tau)+\alpha+\tau+\sigma_k}^{-1} \circ Q_\alpha(S_{\hat{\alpha}})$ should be replaced by $R_{k(2\alpha+\tau)+\alpha+\tau+\sigma_k}^{-1} \circ Q_\alpha(\tilde{S}_{\hat{\alpha},\varepsilon_k,b_k})$ where $\tilde{S}_{\hat{\alpha},\varepsilon_k,b_k}$ is the perturbed hypersphere with scale parameter ε_k and translation parameter b_k. A similar perturbation must also be found for the Clifford torus $\mathbb{T}_\alpha^{n_1,n_2}$. This is provided by the construction used in Butscher and Pacard's articles [5,6] where it was shown how to perturb the Clifford torus in the normal direction using the singular solution of the equation $\mathcal{L}_{n_1,n_2,\alpha}(G) = 0$ on $\mathbb{T}_\alpha^{n_1,n_2} \setminus \{p, -p'\}$. This perturbation is completely analogous to the perturbation of the hyperspheres developed in this article, though it differs in such details as the numerical values of the coefficients of its asymptotic expansions near p and $-p'$. Denote the perturbed Clifford torus by $\tilde{C}_{\alpha,\varepsilon}^{n_1,n_2}$. Then as before, one can find the parameters ε_k, b_k and ε along with neck scale and translation parameters $\bar{\varepsilon}_k$ and $\bar{b}_k$ that ensure optimal matching between these hypersurfaces and small catenoidal necks with the parameters $\bar{\varepsilon}_k, \bar{b}_k$. Moreover, these are all determined uniquely in terms of τ.

Definition 26 Recall the terminology from Sects. 2.3 and 3.2.

- Define the neck regions

$$\mathcal{N}_\varepsilon^{k,\pm} := R_{(\sigma_k+\sigma_{k-1})/2}^{-1} \circ R_{(k+\frac{1}{2})(2\alpha+\tau)+\alpha+\tau}^{-1} \circ Q_\alpha \circ K^{-1}\big(\tilde{\Sigma}_{\bar{\varepsilon}_k,\bar{b}_k}^\pm \cap Cyl(\rho_\varepsilon/2)\big).$$

Set $\mathcal{N}_\varepsilon^k := \mathcal{N}_\varepsilon^{k,+} \cup \mathcal{N}_\varepsilon^{k,-}$. These necks are oriented so that $\mathcal{N}_\varepsilon^{k,+}$ lies behind $\mathcal{N}_\varepsilon^{k,-}$ along the geodesic γ and that the kth neck is centred on the point $R^{-1}_{(\sigma_k+\sigma_{k+1})/2} \circ R^{-1}_{(k+\frac{1}{2})(2\alpha+\tau)+\alpha+\tau}(p)$.

- Define the transition regions

$$\mathcal{T}_\varepsilon^{k,\pm} := R^{-1}_{\sigma_k} \circ R^{-1}_{(k+\frac{1}{2})(2\alpha+\tau)+\alpha+\tau} \circ Q_\alpha \circ K^{-1}\big(\tilde{\Sigma}^\pm_{\bar{\varepsilon}_k,\bar{b}_k} \cap \big[Cyl(2\rho_\varepsilon)\backslash Cyl(\rho_\varepsilon/2)\big]\big).$$

Set $\mathcal{T}_\varepsilon^k := \mathcal{T}_\varepsilon^{k,+} \cup \mathcal{T}_\varepsilon^{k,-}$.

- Define the exterior regions

$$\mathcal{E}_{\alpha,\varepsilon}^{Cliff} := \tilde{C}_{\alpha,\varepsilon_0}^{n_1,n_2}\backslash\big[B_{\tilde{\rho}_{\varepsilon_0}}(p) \cup B_{\tilde{\rho}_{\varepsilon_0}}(-p')\big]$$

$$\mathcal{E}_{\hat{\alpha},\varepsilon}^k := R^{-1}_{\sigma_k} \circ R^{-1}_{k(2\alpha+\tau)+\alpha+\tau} \circ Q_\alpha\big(\tilde{S}_{\alpha,\varepsilon_k,b_k}\backslash\big[B_{\tilde{\rho}_\varepsilon}(p^+) \cup B_{\tilde{\rho}_\varepsilon}(p^-)\big]\big).$$

The approximate solution corresponding to the configuration $\Lambda_{\alpha,\tau,\sigma}^\#$ can now be defined as follows.

Definition 27 The *approximate solution* with parameters α, τ and σ is the hypersurface

$$\tilde{\Lambda}_{\alpha,\tau,\sigma} := \mathcal{E}_{\alpha,\varepsilon}^{Cliff} \cup \left[\bigcup_{k=0}^{N-1}\big(\mathcal{E}_{\hat{\alpha},\varepsilon}^k \cup \mathcal{T}_\varepsilon^k \cup \mathcal{N}_\varepsilon^k\big)\right] \cup \big[\mathcal{T}_\varepsilon^N \cup \mathcal{N}_\varepsilon^N\big]. \tag{49}$$

4.3 Symmetries and Jacobi fields

4.3.1 Symmetries

The generalized Clifford tori are highly symmetric objects in $\mathbb{S}^{n+1}$, being invariant under the diagonal $O(n_1 + 1) \times O(n_2 + 1)$ in $O(n + 2)$. Once the Delaunay-like handle is attached, there is considerably less symmetry. The symmetries which preserve *both* the perturbed generalized Clifford torus and the perturbed hyperspheres are as follows.

1. *Rotational symmetries.*
   Define the transformation $S_{B_1,B_2}^{00} \in O(n + 2)$ by choosing $B_j \in O(n_j + 1)$ and then setting

$$S_{B_1,B_2}^{00} := \begin{pmatrix} 1 & 0 \cdots 0 & 0 & 0 \cdots 0 \\ 0 & & 0 & \\ \vdots & B_1 & \vdots & 0 \\ 0 & & 0 & \\ 0 & 0 \cdots 0 & 1 & 0 \cdots 0 \\ 0 & & 0 & \\ \vdots & 0 & \vdots & B_2 \\ 0 & & 0 & \end{pmatrix} \tag{50}$$

which is the transformation that preserves $\mathbb{T}_\alpha^{n_1,n_2}$ and the geodesic γ as well as the perturbed Clifford torus. These transformations also preserve the perturbed hyperspheres positioned along γ and the necks between them because of the commutation relations $S_{B_1,B_2}^{00} \circ R_\theta = R_\theta \circ S_{B_1,B_2}^{00}$ and

$$Q_\alpha^{-1} \circ S_{B_1,B_2}^{00} \circ Q_\alpha = \begin{pmatrix} 1 & & \\ & 1 & \\ \hline & & B_1 & \\ & & & B_2 \end{pmatrix}$$

which is a transformation that preserves the perturbed hypersphere $\tilde{S}_{\alpha,\varepsilon}$.

2. *Reflection symmetry.*
   One additional symmetry of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ comes from the reflection that reverses the geodesic γ_p with respect to the point half-way between p and $-p'$, namely, the point $R_{\pi-\alpha}(p)$. This is the transformation $T \in O(n+2)$ given by

$$
T := \left(\begin{array}{c|c|c|c}
\begin{array}{c}1 \\ 0 \\ \vdots \\ 0\end{array} & \begin{array}{ccc}0 & \cdots & 0\end{array} & 0 & \begin{array}{ccc}0 & \cdots & 0\end{array} \\
\hline
\begin{array}{c}0 \\ \vdots \\ 0\end{array} & I_{n_1} & \begin{array}{c}0 \\ \vdots \\ 0\end{array} & 0 \\
\hline
0 & \begin{array}{ccc}0 & \cdots & 0\end{array} & -1 & \begin{array}{ccc}0 & \cdots & 0\end{array} \\
\hline
\begin{array}{c}0 \\ \vdots \\ 0\end{array} & 0 & \begin{array}{c}0 \\ \vdots \\ 0\end{array} & I_{n_2}
\end{array}\right).
$$

This reflection preserves the Clifford torus and its perturbation since it is an element of its symmetry group. If the displacement parameters satisfy $\sigma_k = \sigma_{N-k-1}$, then T also preserves the perturbed hyperspheres and the necks because of the commutation relations $T \circ R_\theta \circ T = R_\theta^{-1}$ and

$$
Q_{-\alpha}^{-1} \circ T \circ Q_\alpha = \left(\begin{array}{c|c}
\begin{array}{cc}1 & \\ & -1\end{array} & \\
\hline
 & \begin{array}{cc}I_{n_1} & \\ & I_{n_2}\end{array}
\end{array}\right)
$$

which is a transformation that preserves the perturbed hypersphere $\tilde{S}_{\alpha,\varepsilon}$.

4.3.2 Jacobi fields

It remains to find all the Jacobi fields of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ that are invariant under the transformations S_{B_1,B_2}^{00} and T. As in Sect. 3.4, one proceeds as follows. First define smooth and monotone cut-off functions $\chi_k : \tilde{\Lambda}_{\alpha,\tau,\sigma} \to \mathbb{R}$ for $k = 0, \ldots, N-1$ supported in $\mathcal{E}_\varepsilon^k$ and another such function $\chi_{Cliff} : \tilde{\Lambda}_{\alpha,\tau,\sigma} \to \mathbb{R}$ supported in $\mathcal{E}_{\alpha,\varepsilon}^{Cliff}$. Consider $\tilde{S}_{\hat{\alpha},\varepsilon}$ as a hypersurface in $\mathbb{S}^{n+1} \subseteq \mathbb{R}^{n+2}$ endowed with coordinates $(x^0, x^1, \ldots, x^{n+1})$ and define the following functions:

$$
\tilde{q}_k^t := \chi_k \cdot \left(x^t\big|_{\tilde{S}_{\hat{\alpha},\varepsilon}} \circ Q_\alpha^{-1} \circ R_{k(2\alpha+\tau)+\alpha+\tau} \circ R_{\sigma_k}\right) \tag{51a}
$$

Also, in the standard coordinates (x_1, x_2) for $\mathbb{R}^{n_1+1} \times \mathbb{R}^{n_2+1}$ that are in use at the moment, define the functions

$$
\tilde{q}_{Cliff}^{t_1,t_2} := \chi_{Cliff} \cdot \left(x_1^{t_1} x_2^{t_2}\big|_{\tilde{C}_{\alpha,\varepsilon}^{n_1,n_2}}\right). \tag{51b}
$$

Then the vector space of *all* approximate Jacobi fields in $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ is the set of functions

$$
\tilde{\mathcal{K}} := \text{span}_{\mathbb{R}}\{\tilde{q}_k^t : t = 1, \ldots, n \text{ and all } k\} \cup \{\tilde{q}_{Cliff}^{t_1,t_2} : t_1 = 1, \ldots, n_1 \text{and } t_2 = 1, \ldots, n_2\}.
$$

It is easy to see that the subspace of these functions which are invariant under the symmetries S_{B_1,B_2}^{00} for all $B_j \in O(n_j + 1)$ is spanned by $\{\tilde{q}_k^1 : \text{all } k\} \cup \{\tilde{q}_{Cliff}^{00}\}$. Under the further

requirement of invariance with respect to T, the function $\tilde{q}^{00}_{Cliff}$ is eliminated and one is left with the following subspace.

Definition 28 The subspace of approximate Jacobi fields of $C^{2,\beta}_\delta(\tilde{\Lambda}_{\alpha,\tau,\sigma})$ invariant under the group of symmetries generated by T and $S^{00}_{B_1,B_2}$ for all $B_j \in O(n_j + 1)$ is the subspace of functions

$$\tilde{\mathcal{K}}_{sym} := \operatorname{span}_\mathbb{R}\left\{\tilde{q}^1_k - \tilde{q}^1_{N-k-1} : \text{all } k\right\}. \tag{52}$$

Note that the dimension of $\mathcal{K}_{sym}$ is N.

4.4 The proof of Main Theorem 3

The proof of Main Theorem 3 is now in all respects identical to the proof of Main Theorem 2, for the following reasons. One can set up normal deformations of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ using functions invariant under T and $S^{00}_{B_1,B_2}$ for all $B_j \in O(n_j + 1)$. Then the construction of the right inverse of the linearized mean curvature operator and its estimate follow as in Proposition 21 because the nature of the Jacobi fields of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ is unchanged; namely, there is one invariant Jacobi field on each perturbed hypersphere and none on the necks and on the perturbed Clifford hypersphere by the nature of the symmetries and the choice of weighted function space. This gives the linear estimate (17) needed to invoke the inverse function theorem. Then the estimates of the mean curvature of the Delaunay-like hypersurfaces and the estimates of the mean curvature of the generalized Clifford tori from [5,6] can be combined to yield the first of the non-linear estimates (18) needed to invoke the inverse function theorem. Finally, the second of the non-linear estimates holds as before. Hence when τ is sufficiently small the inverse function theorem can be applied and yields a deformation of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ to a CMC hypersurface that is embedded if and only if $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ is embedded. □

5 Doubling Clifford tori—revisited

5.1 Introduction

The methods developed in this article allow one final example of a CMC hypersurface to be constructed. It is the analogue of the Butscher and Pacard doubling construction for Clifford tori, except for [5,6] Clifford tori of arbitrary mean curvature. That is, one can take two Clifford tori with equal but opposite mean curvature of *any* magnitude, choose a collection of symmetrically located gluing points on each, and place a sequence of hyperspheres of the same mean curvature end-to-end along the geodesics connecting the gluing points on one Clifford torus to those on the other. The entire configuration can then be glued together by inserting catenoidal necks between the hyperspheres, and then perturbed to have exactly CMC.

5.2 The approximate doubling construction

Let $\mathbb{T}^{n_1,n_2}_\alpha$ be the generalized Clifford torus of type $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ with $n_1 + n_2 = n$ and parameter $\alpha \in (0, \pi/2)$ in $\mathbb{S}^{n+1}$. This hypersurface has mean curvature $H^{Cliff}_\alpha := n_2 \cot(\alpha) - n_1 \tan(\alpha)$. Let $\bar{\alpha} \in (0, \pi/2)$ be such that $n_2 \cot(\bar{\alpha}) - n_1 \tan(\bar{\alpha}) = -H^{Cliff}_\alpha$. Note that $\alpha < \alpha_* < \bar{\alpha}$ where

$\alpha_* = \arctan\left(\sqrt{n_2/n_1}\right)$ is such that $H_{\alpha_*}^{Cliff} = 0$. Then the Clifford torus $\mathbb{T}_{\bar{\alpha}}^{n_1,n_2}$ has equal but opposite mean curvature to $\mathbb{T}_{\alpha}^{n_1,n_2}$.

Take a finite subgroup $G \subseteq SO(n+2)$, having order $|G|$, of the type considered in [5,6]. That is, suppose G is a subgroup of the diagonal $O(n_1+1) \times O(n_2+1)$ in $O(n+2)$ and contains the element $T := (T_1, T_2)$ where $T_j \in O(n_j+1)$ is the reflection symmetry across the x_1-axis defined by $T_j(x_1, x_2, \ldots, x_{n_j+1}) := (x_1, -x_2, \ldots, -x_{n_j+1})$. Note that such a group preserves both $\mathbb{T}_{\alpha}^{n_1,n_2}$ and $\mathbb{T}_{\bar{\alpha}}^{n_1,n_2}$. Enumerate the elements of the subgroup G by $G := \{\omega_1 = Id, \omega_2, \ldots, \omega_{|G|}\}$ and denote $\omega_k := (\omega_k^1, \omega_k^2)$ if needed, where $\omega_k^j \in O(n_j+1)$. Now choose the canonical point $p \in \mathbb{T}_{\alpha}^{n_1,n_2}$ defined by $p := (\cos(\alpha)P_1, \sin(\alpha)P_2)$ where $P_j \in \mathbb{R}^{n_j+1}$ is the point $P_j := (1, 0, \ldots, 0)$ and set $p_s := \omega_s(P)$ for every $s = 1, \ldots, |G|$. Similarly, let $\bar{p} := (\cos(\bar{\alpha})P_1, \sin(\bar{\alpha})P_2) \in \mathbb{T}_{\bar{\alpha}}^{n_1,n_2}$ and set $\bar{p}_s := \omega_s(\bar{P})$ for every $s = 1, \ldots, |G|$. Assume that $\omega_1 = Id$ so that $p_1 = p$ and $\bar{p}_1 = \bar{p}$. Let M be the orbit of p under G and let $\bar{M}$ be the orbit of $\bar{p}$ under G. Finally, let γ_s be the geodesic taking p_s to $\bar{p}_s$. Note that γ_s is normal to $\mathbb{T}_{\alpha}^{n_1,n_2}$ at p_s, normal to $\mathbb{T}_{\bar{\alpha}}^{n_1,n_2}$ at $\bar{p}_s$ and can be parametrized by $t \longmapsto \omega_s \circ R_t(p)$ where $R_t \in SO(n+2)$ is the rotation introduced in (46).

The next task is to position hyperspheres between $\mathbb{T}_{\alpha}^{n_1,n_2}$ and $\mathbb{T}_{\bar{\alpha}}^{n_1,n_2}$ along each γ_s. Recall that the mean curvature of a hypersphere $S_{\hat{\alpha}}$ with parameter $\hat{\alpha} \in (0, \pi/2)$ is $H_{\hat{\alpha}}^{Sph} := n\cot(\hat{\alpha})$. Therefore the hypersphere with the same mean curvature (up to sign) as $\mathbb{T}_{\alpha}^{n_1,n_2}$ is the one with $\hat{\alpha} := \text{arccot}\left|\frac{n_2}{n}\cot(\alpha) - \frac{n_1}{n}\tan(\alpha)\right|$. Suppose now that the parameters α, $\bar{\alpha}$ and $\hat{\alpha}$ are such that there are positive integers N and m and a small separation parameter $0 < \tau \ll 2\pi$ so that $2N\hat{\alpha} + (N+1)\tau = \bar{\alpha} - \alpha + 2m\pi$. It is thus possible to position N rotated hyperspheres of parameter $\hat{\alpha}$ along each γ_s from p_s to $\bar{p}_s$, where each hypersphere is separated from its two nearest neighbours by a distance τ. These spheres wind m times around γ_s before ending up a distance of exactly τ from $\bar{p}_s$.

Remark It is clear that for every fixed $\alpha \in (0, \pi/2)$ and for every $\tau_0 > 0$ there is always *some* value of τ satisfying $0 < \tau < \tau_0$ and *some* integers N and m so that the equation $2N\hat{\alpha} + (N+1)\tau = \bar{\alpha} - \alpha + 2m\pi$ holds. It is not clear if this equation can be made to hold in this way for fixed m as well (e.g., for $m = 0$ which would correspond to embedded configurations). But if α is sufficiently close to 0 and thus $\bar{\alpha}$ is sufficiently close to $\pi/2$ then as before there is a sequence of examples having increasingly smaller τ satisfying the condition with $m = 0$.

These considerations lead to the definition of the initial configuration of hyperspheres and Clifford tori that will be glued together below. Recall the rotation $Q_\alpha \in SO(n+2)$ introduced in (47) which takes the point $p_0 := (1, 0, \ldots, 0) \in \mathbb{R}^{n+2}$ to the point p, the vector $N_0 := (0, 1, 0, \ldots 0) \in \mathbb{R}^{n+2}$ to the vector N_p, and the subspace $\{0, 0\} \times \mathbb{R}^n$ to the tangent space of $\mathbb{T}_{\alpha}^{n_1,n_2}$ at p. Let $\sigma_1, \ldots, \sigma_N$ be small displacement parameters and define the initial configuration

$$\Lambda_{\alpha,\tau,\sigma}^{\#} := \mathbb{T}_{\alpha}^{n_1,n_2} \cup \left[\bigcup_{s=1}^{|G|} \bigcup_{k=1}^{N} \omega_s \circ R_{(2k-1)\hat{\alpha}+k\tau+\sigma_k} \circ Q_\alpha(S_{\hat{\alpha}})\right] \cup \mathbb{T}_{\bar{\alpha}}^{n_1,n_2}. \tag{53}$$

Note that there is redundancy in the labelling of the spheres in (53) corresponding to the subgroup of G fixing the point p. When ω_s fixes p then $\omega_s = S_{B_1,B_2}^{00}$ for some $B_j \in O(n_j+1)$ as defined in (50). Thus $\omega_s \circ R_{(2k-1)\hat{\alpha}+k\tau+\sigma_k} \circ Q_\alpha(S_{\hat{\alpha}}) = R_{(2k-1)\hat{\alpha}+k\tau+\sigma_k} \circ Q_\alpha \circ R(S_{\hat{\alpha}})$ where R is a transformation preserving both the sphere $S_{\hat{\alpha}}$ and the $\{x^0, x^1\}$-equator. Consequently, the entire initial configuration $\Lambda_{\alpha,\tau,\sigma}^{\#}$ is also invariant under G.

To glue the initial configuration together, one proceeds as follows. By analogy with the procedure described in Sect. 4.2, one first replaces each hypersphere $\omega_s \circ R_{(2k-1)\alpha + k\tau + \sigma_k} \circ Q_\alpha(S_{\hat{\alpha}})$ with $\omega_s \circ R_{(2k-1)\alpha + k\tau + \sigma_k} \circ Q_\alpha(\tilde{S}_{\hat{\alpha}, \varepsilon_k, b_k})$ where $\tilde{S}_{\hat{\alpha}, \varepsilon_k, b_k}$ is the perturbed hypersphere of Sect. 3.2 and the parameters ε_k, b_k are the usual scale and translation parameters. Then one replaces the Clifford tori $\mathbb{T}_\alpha^{n_1, n_2}$ and $\mathbb{T}_{\tilde{\alpha}}^{n_1, n_2}$ with the perturbed hypersurfaces used in Butscher and Pacard's articles [5,6]. Denote these $\tilde{C}_{\alpha, \varepsilon}^{n_1, n_2}$ and $\tilde{C}_{\tilde{\alpha}, \varepsilon}^{n_1, n_2}$ corresponding to $\mathbb{T}_\alpha^{n_1, n_2}$ and $\mathbb{T}_{\tilde{\alpha}}^{n_1, n_2}$, respectively. Then as before, one can find the parameters ε_k, b_k and ε along with neck scale and translation parameters $\bar{\varepsilon}_k$ and $\bar{b}_k$ that ensure optimal matching between these hypersurfaces and small catenoidal necks with the parameters $\bar{\varepsilon}_k$, $\bar{b}_k$. Moreover, these are uniquely determined in terms of τ.

Definition 29 Recall the terminology from Sects. 2.3, 3.2 and 4.2.

- Define the neck regions

$$\mathcal{N}_\varepsilon^{sk, \pm} := \omega_s \circ R_{2k\hat{\alpha} + (k+1/2)\tau + (\sigma_k + \sigma_{k+1})/2} \circ Q_\alpha \circ K^{-1}\big(\tilde{\Sigma}_{\bar{\varepsilon}_k, \bar{b}_k}^\pm \cap Cyl(\rho_\varepsilon/2)\big).$$

Set $\mathcal{N}_\varepsilon^{sk} := \mathcal{N}_\varepsilon^{sk, +} \cup \mathcal{N}_\varepsilon^{sk, -}$. These necks are oriented so that $\mathcal{N}_\varepsilon^{sk, +}$ lies ahead of $\mathcal{N}_\varepsilon^{sk, -}$ along the geodesic γ_s and that the kth neck is centred on the point $\omega_s \circ R_{2k\hat{\alpha} + (k+1/2)\tau + (\sigma_k + \sigma_{k+1})/2}(p)$.

- Define the transition regions

$$\mathcal{T}_\varepsilon^{sk, \pm} := \omega_s \circ R_{2k\hat{\alpha} + (k+1/2)\tau + (\sigma_k + \sigma_{k+1})/2} \circ Q_\alpha \circ K^{-1}\big(\tilde{\Sigma}_{\bar{\varepsilon}_k, \bar{b}_k}^\pm \cap \big[Cyl(2\rho_\varepsilon) \backslash Cyl(\rho_\varepsilon/2)\big]\big).$$

Set $\mathcal{T}_\varepsilon^{sk} := \mathcal{T}_\varepsilon^{sk, +} \cup \mathcal{T}_\varepsilon^{sk, -}$.

- Define the exterior regions

$$\mathcal{E}_{\alpha, \varepsilon}^{Cliff} := \tilde{C}_{\alpha, \varepsilon_0}^{n_1, n_2} \backslash \big[B_{\tilde{\rho}_{\varepsilon_0}}(p) \cup B_{\tilde{\rho}_{\varepsilon_0}}(-p')\big]$$

$$\mathcal{E}_{\tilde{\alpha}, \varepsilon}^{Cliff} := \tilde{C}_{\tilde{\alpha}, \varepsilon_0}^{n_1, n_2} \backslash \big[B_{\tilde{\rho}_{\varepsilon_0}}(p) \cup B_{\tilde{\rho}_{\varepsilon_0}}(-p')\big]$$

$$\mathcal{E}_{\hat{\alpha}, \varepsilon}^{sk} := \omega_s \circ R_{(2k-1)\hat{\alpha} + k\tau + \sigma_k} \circ Q_\alpha\big(\tilde{S}_{\alpha, \varepsilon_k, b_k} \backslash \big[B_{\tilde{\rho}_\varepsilon}(p^+) \cup B_{\tilde{\rho}_\varepsilon}(p^-)\big]\big).$$

The approximate solution corresponding to the configuration $\Lambda_{\alpha, \tau, \sigma}^\#$ can now be defined as follows.

Definition 30 The *approximate solution* with parameters α, τ and σ is the hypersurface

$$\tilde{\Lambda}_{\alpha, \tau, \sigma} := \mathcal{E}_{\alpha, \varepsilon}^{Cliff} \cup \big[\mathcal{T}_\varepsilon^{s0} \cup \mathcal{N}_\varepsilon^{s0}\big] \cup \left[\bigcup_{s=1}^{|G|} \bigcup_{k=1}^{N} \big(\mathcal{E}_{\hat{\alpha}, \varepsilon}^{sk} \cup \mathcal{T}_\varepsilon^{sk} \cup \mathcal{N}_\varepsilon^{sk}\big)\right] \cup \mathcal{E}_{\tilde{\alpha}, \varepsilon}^{Cliff}. \qquad (54)$$

5.3 Symmetries and Jacobi fields

The initial configuration $\Lambda_{\alpha, \tau, \sigma}^\#$ is G-invariant and it is clear that the gluing procedure leading up to the definition of the approximate solution $\tilde{\Lambda}_{\alpha, \tau, \sigma}$ can also be done in a G-invariant manner. It remains to see which of the Jacobi fields of $\tilde{\Lambda}_{\alpha, \tau, \sigma}$ are G-invariant. As before, one proceeds as follows. First define a smooth and monotone cut-off function $\chi_{sk} : \tilde{\Lambda}_{\alpha, \tau, \sigma} \to \mathbb{R}$ supported in $\mathcal{E}_{\hat{\alpha}, \varepsilon}^{sk}$ and two other such functions $\chi_{\alpha, Cliff} : \tilde{\Lambda}_{\alpha, \tau, \sigma} \to \mathbb{R}$ supported in $\mathcal{E}_{\alpha, \varepsilon}^{Cliff}$ and $\chi_{\tilde{\alpha}, Cliff} : \tilde{\Lambda}_{\alpha, \tau, \sigma} \to \mathbb{R}$ supported in $\mathcal{E}_{\tilde{\alpha}, \varepsilon}^{Cliff}$. Without loss of generality, the functions $\chi_{\alpha, Cliff}$

and $\chi_{\bar{\alpha}, Cliff}$ are G-invariant while $\chi_{sk} \circ \omega_s \circ \omega_t^{-1} = \chi_{tk}$ holds for all $s, t = 1, \ldots, |G|$. Consider $\tilde{S}_{\hat{\alpha}, \varepsilon}$ as a hypersurface in $\mathbb{S}^{n+1} \subseteq \mathbb{R}^{n+2}$ endowed with coordinates $(x^0, x^1, \ldots, x^{n+1})$ and define the following functions.

$$\tilde{q}_{sk}^t := \chi_{sk} \cdot \left(x^t \big|_{\tilde{S}_{\hat{\alpha}, \varepsilon}} \circ Q_{\alpha}^{-1} \circ R_{(2k-1)\alpha + k\tau + \sigma_k}^{-1} \circ \omega_s^{-1} \right). \tag{55a}$$

Also, in the standard coordinates (x_1, x_2) for $\mathbb{R}^{n_1+1} \times \mathbb{R}^{n_2+1}$ that are in use at the moment, define the functions

$$\begin{aligned}
\tilde{q}_{\alpha, Cliff}^{t_1, t_2} &:= \chi_{\alpha, Cliff} \cdot \left(x_1^{t_1} x_2^{t_2} \big|_{\tilde{C}_{\alpha, \varepsilon}^{n_1, n_2}} \right) \\
\tilde{q}_{\bar{\alpha}, Cliff}^{t_1, t_2} &:= \chi_{\bar{\alpha}, Cliff} \cdot \left(x_1^{t_1} x_2^{t_2} \big|_{\tilde{C}_{\bar{\alpha}, \varepsilon}^{n_1, n_2}} \right).
\end{aligned} \tag{55b}$$

Then the vector space of *all* approximate Jacobi fields in $C^{2,\beta}(\tilde{\Lambda}_{\alpha, \tau, \sigma})$ is the set of functions

$$\tilde{\mathcal{K}} := \text{span}_{\mathbb{R}} \bigcup_{s=1}^{|G|} \bigcup_{k=1}^{N} \{\tilde{q}_{sk}^t : t = 1, \ldots, n\}$$
$$\cup \{\tilde{q}_{\alpha, Cliff}^{t_1, t_2}, \tilde{q}_{\bar{\alpha}, Cliff}^{t_1, t_2} : t_1 = 1, \ldots, n_1 \text{ and } t_2 = 1, \ldots, n_2\}.$$

At this point, impose the condition on the group G used in Butscher and Pacard's doubling construction [5,6] that there is no linear combination of the $\tilde{q}_{\alpha, Cliff}^{t_1, t_2}$ and $\tilde{q}_{\bar{\alpha}, Cliff}^{t_1, t_2}$ that is invariant under every element of G. This eliminates the Jacobi fields on the perturbed Clifford tori from consideration when only equivariant perturbations are allowed. Furthermore, invariance with respect to G implies first of all that the Jacobi fields on the kth perturbed hypersphere along every geodesic are the same. Then invariance with respect to the reflection $T \in G$ implies that the only invariant Jacobi fields of $\tilde{\Lambda}_{\alpha, \tau, \sigma}$ are to be found in the following subspace.

Definition 31 The subspace of approximate Jacobi fields of $\tilde{\Lambda}_{\alpha, \tau, \sigma}$ invariant under G is the subspace of functions

$$\tilde{\mathcal{K}}_{sym} := \text{span}_{\mathbb{R}} \left\{ \sum_{s=1}^{|G|} \tilde{q}_{sk}^1 : k = 1, \ldots, N \right\}. \tag{56}$$

Note that the dimension of $\mathcal{K}_{sym}$ is N.

5.4 The proof of Main Theorem 4

The proof of Main Theorem 4 is now in all respects identical to the proof of Main Theorem 3, for the following reasons. One can set up normal deformations of $\tilde{\Lambda}_{\alpha, \tau, \sigma}$ using functions invariant under G. Then the construction of the right inverse of the linearized mean curvature operator and its estimate follow as in Proposition 21 and the analogous constructions performed in Butscher and Pacard's articles [5,6] because the nature of the Jacobi fields of $\tilde{\Lambda}_{\alpha, \tau, \sigma}$ is unchanged; namely, there is one invariant Jacobi field on each perturbed hypersphere and none on the necks and on the perturbed Clifford hyperspheres by the nature of the symmetries and the choice of weighted function space. This gives the linear estimate (17) needed to invoke the inverse function theorem. Then the estimates of the mean curvature of the Delaunay-like hypersurfaces and the estimates of the mean curvature of the generalized Clifford tori from [5,6] can again be combined to yield the first of the non-linear estimates

([18](#)) needed to invoke the inverse function theorem. Finally, the second of the non-linear estimates holds as before. Hence when τ is sufficiently small the inverse function theorem can be applied and yields a deformation of $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ to a CMC hypersurface that is embedded if and only if $\tilde{\Lambda}_{\alpha,\tau,\sigma}$ is embedded. $\qquad\square$

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