Equivariant gluing constructions of contact stationary Legendrian submanifolds in $\mathbb{S}^{2n+1}$

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Abstract   A contact-stationary Legendrian submanifold of $\mathbb{S}^{2n+1}$ is a Legendrian submanifold whose volume is stationary under contact deformations. The simplest contact-stationary Legendrian submanifold (actually minimal Legendrian) is the real, equatorial n-sphere S_0. This paper develops a method for constructing contact-stationary (but not minimal) Legendrian submanifolds of $\mathbb{S}^{2n+1}$ by gluing together configurations of sufficiently many $U(n+1)$-rotated copies of S_0. Two examples of the construction, corresponding to finite cyclic subgroups of $U(n+1)$ are given. The resulting submanifolds are very symmetric; are geometrically akin to a 'necklace' of copies of S_0 attached to each other by narrow necks and winding a large number of times around $\mathbb{S}^{2n+1}$ before closing up on themselves; and are topologically equivalent to $\mathbb{S}^1 \times \mathbb{S}^{n-1}$.

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1 Introduction and statement of results

1.1 Background

Lagrangian variational problems It has been known for some time that minimal and Lagrangian submanifolds of a M possess a rich mathematical structure. In addition, it is possible to pose two very natural *restricted* variational problems in the class of Lagrangian submanifolds of M whose critical points are also mathematically quite interesting. First, one can demand that the volume of a Lagrangian submanifold L is stationary with respect to all variations of L which preserve the Lagrangian condition. A natural sub-class of variations preserving the Lagrangian condition is the set of *Hamiltonian transformations* generated by functions of M. Therefore one can also demand that the volume of L is stationary with respect to these variations. In the former case, L is said to be *Lagrangian stationary*; in the latter case, L is said to be *Hamiltonian stationary*.

The stationarity of Lagrangian submanifolds of a Kähler–Einstein manifold M under Lagrangian and Hamiltonian deformations has been studied by several authors, notably Oh [21,22], Hélein and Romon [11–13], Schoen and Wolfson [25,26]. Oh initially posed the Lagrangian and Hamiltonian stationary variational problems and derived first and second variation formulæ. Hélein and Romon showed that when M is a two-complex-dimensional Hermitian symmetric space, this stationarity condition can be reformulated as an infinite-dimensional integrable system whose solutions possess a Weierstraß-type representation similar to the Dorfmeister–Pedit–Wu representation for harmonic surfaces in a real symmetric space [5]. Moreover, they found all Hamiltonian stationary, doubly periodic immersions of $\mathbb{R}^2$ into $\mathbb{CP}^2$ using this representation. They did not address the question of when these doubly periodic immersions close up to form compact immersions or embeddings, though the existence of such configurations is certainly expected. Finally, Schoen and Wolfson initiated the study of the Lagrangian variational problem from the geometric analysis point of view, treating it as a method for constructing minimal and Lagrangian submanifolds, perhaps with singularities, as limits of volume-minimizing sequences of Lagrangian submanifolds. The rationale behind this approach is their observation that a Lagrangian stationary submanifold of M that is smooth must necessarily be minimal (because the mean curvature vector field of L is itself the infinitesimal generator of a Lagrangian variation).

Singularities and Legendrian variational problems From the work of Schoen and Wolfson it has emerged that questions concerning the regularity of limits of volume-minimizing sequences of Lagrangian submanifolds are very delicate and that the structure of the singularities of Lagrangian stationary submanifolds can be very complicated. Indeed, general structure theorems for these singularities (such as classifications) are currently well out of reach and the most fruitful approach at the moment is the constructions of classes

of examples of Lagrangian stationary, Hamiltonian stationary, and minimal Lagrangian submanifolds possessing singularities of various types.

A class of singularities that is particularly amenable to study is when the stationary Lagrangian submanifold possesses an *isolated conical singularity* having *multiplicity one*. Such a singularity occurs when the stationary Lagrangian submanifold L is singular at an isolated point where the tangent cone (in the sense of geometric measure theory) is modeled on an actual cone, i.e. the tangent cone C is a complete, homothetically invariant, submanifold of Euclidean space. In this case, C is determined by the submanifold Λ formed by intersecting C with the unit sphere in Euclidean space, called the *link* of the cone. The Lagrangian stationary, Hamiltonian stationary, and minimal Lagrangian conditions then translate into conditions satisfied by Λ. The fundamental observations are: if a Lagrangian cone C is minimal then Λ is a *Legendrian submanifold* with respect to the *contact structure* of the sphere that is itself minimal; and if C is Lagrangian or Hamiltonian stationary, then Λ is stationary under all variations of Λ through Legendrian submanifolds. It turns out that this is equivalent to Λ being stationary under all contact structure-preserving variations (a.k.a *contactomorphisms*) and Λ is said to be *contact-stationary*.

Minimal and contact-stationary Legendrian submanifolds of $\mathbb{S}^{2n+1}$ The study of minimal Legendrian and contact-stationary Legendrian submanifolds of the sphere $\mathbb{S}^{2n+1}$ is relatively recent endeavour, but a certain number of results exist. The simplest case is in dimension $n = 1$. It is clear that the minimal Legendrian submanifolds are contact curves that are also great circles. The contact-stationary submanifolds are non-trivial, however. They are the so-called (p, q)-curves discovered by Schoen and Wolfson in [26], where p and q are relatively prime integers.

In higher dimensions, the situation changes entirely. In dimension $n = 2$, Yau showed in [28,29] that genus zero minimal Legendrian submanifolds of $\mathbb{S}^5$ are trivial; i.e. they are equatorial two spheres. Moreover, there are no contact-stationary Legendrian submanifolds of genus zero that are not minimal; as will be further explained later, this is because 2-spheres possess no non-trivial harmonic one-forms. In higher genus, however, the situation is quite different. In genus one, a huge abundance of minimal Legendrian tori in $\mathbb{S}^5$ has recently been discovered that can be studied using techniques of integrable systems theory [4,8,20] and work has been done to understand the so-called 'geometric complexity' of these torus cones, in the sense of Haskins [8]. This latter task is important since Joyce has conjectured that the singular minimal Lagrangian submanifolds of $\mathbb{C}^3$ with isolated conical singularities whose cone links have the least geometric complexity are those which occur 'most often' in the boundary of the moduli space of all minimal Lagrangian submanifolds [14]. There are plenty of contact-stationary, doubly periodic Legendrian immersions of $\mathbb{R}^2$ into $\mathbb{S}^5$: all are lifts from $\mathbb{C}P^2$ of Hélein and Romon's Hamiltonian stationary examples. But the question of which of these close up to form tori is not known explicitly, except for the *homogeneous* tori classified by Hélein and Romon. In higher genus, Wang [27] has constructed minimal Legendrian submanifolds of various genera in $\mathbb{S}^5$ using a reflection technique, but these are not everywhere smooth. The question of whether *smooth* and *embedded* higher-genus minimal Legendrian submanifolds exist in $\mathbb{S}^5$ has only been settled very recently by Haskins and Kapouleas [10]. These authors have constructed odd-genus minimal Legendrian submanifolds by fusing together many copies of Haskins' $U(1)$-invariant minimal Legendrian tori [9] in a manner analogous to Kapouleas' fusion of Wente tori [15]. The question of the existence of higher-genus contact-stationary Legendrian submanifolds of $\mathbb{S}^5$ is open, though in light of Haskins' and Kapouleas' result, the answer is most certainly that these do exist. Furthermore, it is expected that the Haskins' and Kapouleas result can be generalized to higher dimensions.

The purpose of this paper is to develop a method of constructing contact-stationary Legendrian submanifolds of $\mathbb{S}^{2n+1}$ using a gluing technique starting from the simplest building blocks. That is, an approximately contact-stationary Legendrian submanifold will be constructed by forming connected sums of simple, minimal Legendrian building blocks; and then this construction will be perturbed (by solving the non-linear PDE satisfied by contact-stationary Legendrian submanifolds) to yield an exactly contact-stationary Legendrian submanifold. The method is such that for cohomological reasons, the contact-stationary Legendrian submanifold that results from the perturbation process cannot be minimal.

1.2 The Main Theorem

Preliminaries   In order to state the Main Theorem to be proved in this paper, it is necessary to introduce some important terminology. The first concept that must be put into words is a special way in which two Legendrian submanifolds of the sphere can intersect each other.

Definition 1 *Let* Λ, Λ' *be two Legendrian submanifolds of* $\mathbb{S}^{2n+1}$. *Then* Λ *and* Λ' *are* contact-transverse *over a Hopf fiber* $\mathcal{F}$ *if the following two conditions are satisfied.*

1.   *There exists a point* $p \in \Lambda \cap \mathcal{F}$ *and a complex number* e^{ic} *so that* $\mathrm{e}^{ic} p \in \Lambda' \cap \mathcal{F}$.
2. $T_p \Lambda \oplus T_p(\mathrm{e}^{-ic} \Lambda') = \Xi_p$.

The submanifolds Λ *and* Λ' *are said to have* contact-transverse intersection *at* p *if* $c = 0$ *in the definition above, so that* $p \in \Lambda \cap \Lambda'$.

Definition 2 *If* Λ *and* Λ' *are contact-transverse over a Hopf fiber* $\mathcal{F}$, *then the number* c *is called the* Hopf separation *between* Λ *and* Λ' *over* $\mathcal{F}$.

A construction needed in this paper is the *Legendrian connected sum* of two Legendrian submanifolds Λ and Λ' with contact-transverse intersection at a point p. The details of this construction will be given in Sect. 3.4. In the mean time, it suffices to state the condition on the tangent spaces $T_p \Lambda$ and $T_p \Lambda'$ under which there will exist a Legendrian neck that can be used to connect Λ to $\mathrm{e}^{i\alpha} \Lambda'$ in a neighbourhood of p.

The condition can be phrased in terms of an *angle criterion* satisfied by the *characteristic angles* of the direct sum of these two tangent spaces. First, realize that Ξ_p is isomorphic to the symplectic vector space $\mathbb{C}^n$, and $T_p \Lambda$ and $T_p \Lambda'$ are isomorphic to Lagrangian n-planes in $\mathbb{C}^n$. It can be shown that for any pair of transversely intersecting Lagrangian n-planes Π_1 and Π_2 in $\mathbb{C}^n$, there exists a Hermitian orthonormal basis $E_1, \ldots, E_n$ of $\mathbb{C}^n$ and a unique set of angles $\theta_k \in (0, \pi/2]$ for $k = 1, \ldots, n-1$ and $\theta_n \in [\pi/2, \pi)$ such that

$$\Pi_1 = \mathrm{span}_{\mathbb{R}} \{E_1, \ldots, E_n\} \quad \text{and} \quad \Pi_2 = \mathrm{span}_{\mathbb{R}} \left\{\mathrm{e}^{i\theta_1} E_1, \ldots, \mathrm{e}^{i\theta_n} E_n\right\}.$$

The existence of this basis follows from standard complex linear algebra and can be found in [6].

Definition 3 *The* angle criterion *for the pair of n-planes* Π_1 *and* Π_2 *is that their characteristic angles* $\theta_1, \ldots, \theta_n$ *satisfy* $\theta_1 + \cdots + \theta_n = \pi$.

Note that the angle criterion need not hold for a general pair of intersecting special Lagrangian planes, though it is *always* satisfied in dimensions $n = 2, 3$ for numerical reasons.

The concept behind the construction The idea behind the construction of this paper is to use $U(n + 1)$-rotated copies of the real equatorial n-sphere $S_0 = \mathbb{S}^{2n+1} \cap (\mathbb{R}^{n+1} \times \{0\})$ as building blocks that can be connected together at points of contact-transverse intersection by means of small Legendrian necks to form a long, closed chain of n-spheres having the topology of the closed cylinder $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ and winding around $\mathbb{S}^{2n+1}$ a number of times. The construction of the necks will also be carried out herein and is based on the Lawlor neck [16,17] which has already been used in connected sum constructions involving intersecting minimal Lagrangian submanifolds [3,18,19].

Since each n-sphere is minimal Legendrian, the first hope is that the long chain will be almost *minimal* Legendrian and can be perturbed using a small contactomorphism into an exactly minimal Legendrian submanifold. However, there is a good reason why this should not be possible. To see this, first recall that $\mathbb{C}^{n+1}$ carries a parallel, non-vanishing holomorphic volume form $\Omega = \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^{n+1}$. If L is a Lagrangian submanifold then it can be shown that $\Omega(E_1, \ldots E_{n+1})$ is a complex number of unit length for every orthonormal basis $E_1, \ldots, E_{n+1}$ for $T_p L$ and $p \in L$. Furthermore, the value of this complex number is independent of the choice of basis so that the prescription $p \mapsto \exp(\mathrm{i}\Theta_L(p))$ defines a possibly multi-valued function Θ_L on L which is called the *Lagrangian angle function* [7]. It can also be shown that this function satisfies $H_L \lrcorner \, \omega\big|_L = \mathrm{d}\Theta_L$ where H_L is the mean curvature vector of L. Hence if L is minimal and Lagrangian then $\Theta_L = constant$ and if L is a Hamiltonian stationary submanifold, then $\Delta\Theta_L = 0$ locally and $\mathrm{d}\Theta_L$ defines a harmonic one-form globally. Finally, if $L = C(\Lambda)$ then the Lagrangian angle function is also determined by its values on Λ, so that one can define a *Legendrian angle function* Θ_Λ, which is constant when Λ is minimal Legendrian and defines a harmonic one-form when Λ is contact-stationary.

It turns out that the Legendrian necks have a non-negligible *height* in the Hopf fiber direction. Thus to use such a neck for connecting two spheres with the least error, it is first necessary to create just the right amount of Hopf separation between them. The problem is that the Legendrian angle function of a translate of S_0 by $e^{\mathrm{i}\alpha}$ along the Hopf fibers is α. Thus if a large number of n-spheres is attached together and the angle function increases by α from one sphere to the next, then the angle function can not remain small. Moreover, if the long chain of n-spheres winds completely around $\mathbb{S}^{2n+1}$ and closes up, then the Legendrian angle function must acquire a period and cease to be single-valued. One would therefore not expect to be able to deform such a configuration into a Legendrian submanifold with angle zero.

The solution is to give up attempting to construct a minimal Legendrian chain of n-spheres and instead attempt to construct a contact-stationary Legendrian chain of n-spheres. The angle function of such an object must satisfy $\nabla \cdot \mathrm{d}\Theta_\Lambda = 0$ so a slowly varying angle function acquiring a period around the chain can be incorporated into the construction. And since there are cohomologically non-trivial harmonic one-forms on a closed cylinder, one might expect to find solutions.

The Main Theorem The Main Theorem of this paper is that the concept posed in the previous paragraphs can be realized provided the building block n-spheres and Legendrian necks can be assembled in a sufficiently symmetric manner. The first step is to construct an *approximate solution* of the problem by gluing together the union of spheres $\bigcup_s (\zeta U)^s (S_0)$ where $U \in SU(n + 1)$ is a suitably chosen generator of a cyclic subgroup of $SU(n + 1)$ of integer order N and $\zeta := e^{2\pi \mathrm{i}a/kN}$ for some integer a and large integer k satisfying g.c.d.$(a, kN) = 1$ whose purpose is to create small Hopf separation between the copies of S_0 at special points. The resulting submanifold will be denoted by $\tilde{\Lambda}_{U,\zeta}$ and will be smooth,

Legendrian, and if U and ζ are chosen properly, also embedded. Each $(\zeta U)^s(S_0)$ is connected to exactly two of its neighbours in such a way that what is produced is a chain of kN rotated copies of S_0 winding a times around $\mathbb{S}^{2n+1}$ that eventually closes up. The Legendrian angle function on the part of $(\zeta U)^s(S_0)$ away from the necks is almost constant equal to $2\pi sa/kN$. Hence that angle function increases only in the neck regions from one n-sphere to the next; and increases by $2\pi a$ for every loop around the chain.

The means for deforming $\tilde{\Lambda}_{U,\zeta}$ into a contact-stationary Legendrian submanifold will be to generate small contact deformations from functions defined on $\tilde{\Lambda}_{U,\zeta}$. Without going into the details, let $\phi_f : \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}$ denote the contactomorphism generated by the function $f : \tilde{\Lambda}_{U,\zeta} \to \mathbb{R}$. The function f will then be selected to satisfy the Euler–Lagrange equation satisfied by contact-stationary Legendrian submanifolds, which will turn out to be $\nabla \cdot \left(H_{\phi_f(\tilde{\Lambda}_{U,\zeta})} \lrcorner \, \mathrm{d}\alpha \right) = 0$, a fourth order, nonlinear, elliptic partial differential equation on $\tilde{\Lambda}_{U,\zeta}$. This equation will be solved perturbatively near zero; and doing so hinges on being able to find a right inverse, bounded above by a constant independent of k, for the linearization of this equation at zero, hereinafter denoted $\mathcal{L}_{U,\zeta}$. However, it is a manifestation of a general phenomenon in geometric singular perturbation problems that $\mathcal{L}_{U,\zeta}$ can have small eigenvalues tending to zero as $k \to \infty$. The associated eigenfunctions are called *Jacobi fields* and the associated co-kernel constitutes an obstruction to solvability.

One way to avoid the obstruction described above is to exploit symmetry. That is, if $\tilde{\Lambda}_{U,\zeta}$ possesses a group of isometries, then one can deform it equivariantly; i.e. by contactomorphisms that are invariant under the isometries. It will turn out that, if U is chosen correctly, the group of isometries is so large that there are no Jacobi fields invariant under all isometries of $\tilde{\Lambda}_{U,\zeta}$ at once, so that equivariant deformation has the effect of eliminating these Jacobi fields. The theorem that will be proved in this way is as follows.

Main Theorem *Let $\tilde{\Lambda}_{U,\zeta}$ be the approximately contact-stationary Legendrian submanifold of $\mathbb{S}^{2n+1}$ constructed above. If U is chosen appropriately, then there exists k_0 so that if $k > k_0$, then there is a small contact deformation of $\tilde{\Lambda}_{U,\zeta}$ into a contact-stationary Legendrian submanifold $\Lambda_{U,\zeta}$ possessing the same symmetries as $\tilde{\Lambda}_{U,\zeta}$. Finally, $\Lambda_{U,\zeta}$ is embedded whenever $\tilde{\Lambda}_{U,\zeta}$ is.*

Two examples of $SU(n+1)$-rotations to which the theorem applies will be given in Sect. 4.1.

2 Geometric preliminaries

2.1 Riemannian contact manifolds

Contact structures A contact structure on an odd-dimensional manifold M^{2n+1} is a hyperplane field Ξ on M, which is fully non-integrable in the sense of the Frobenius theorem. A 1-form α on M whose kernel at $p \in M$ is the contact hyperplane $\Xi_p \subseteq T_pM$ is called a *contact form* and full non-integrability can be expressed via the requirement that $\alpha \wedge (\mathrm{d}\alpha)^{\wedge n} \neq 0$. In turn, this relationship ensures that $\mathrm{d}\alpha$ is a non-degenerate, skew-symmetric two-form on each contact hyperplane—in short, a symplectic form. The local structure theorem in contact geometry is *Gray's Theorem*: it asserts that every contact manifold is locally diffeomorphic to $\mathbb{R}^{2n} \times \mathbb{R}$ with coordinates (x, y, t) and a 'standard' contact form such as

$$\alpha := \mathrm{d}t + \frac{1}{2}\sum_{k=1}^{n}\left(x^k\mathrm{d}y^k - y^k\mathrm{d}x^k\right) \quad \text{or} \quad \alpha' := \mathrm{d}t - \sum_{k=1}^{n}y^k\mathrm{d}x^k.$$

Note that $\alpha\left(\frac{\partial}{\partial t}\right) = \alpha'\left(\frac{\partial}{\partial t}\right) = 1$ and $\mathrm{d}\alpha = \mathrm{d}\alpha' = \sum_{k=1}^{n} \mathrm{d}x^k \wedge \mathrm{d}y^k$ is the standard symplectic form of $\mathbb{R}^{2n}$. These are the defining features of a 'standard' contact structure. Gray's Theorem is the analogue of the Darboux Theorem of symplectic geometry.

A contact manifold M possesses a canonical non-vanishing vector field called the *Reeb vector field* defined by the requirement that $R \lrcorner \, \mathrm{d}\alpha = 0$ and $\alpha(R) = 1$. For example, the standard contact structures of $\mathbb{R}^{2n+1}$ have Reeb vector field $R := \frac{\partial}{\partial t}$. Denote by $[p]$ the integral curve of the Reeb vector field passing through p. The set of all integral curves of the Reeb field is called the *characteristic foliation*. When $[p]$ is a smooth, one-dimensional embedded submanifold for every $p \in M$, then the space of fibers is a smooth manifold $\hat{M} = \{[p] : p \in M\}$. The canonical projection $\pi : M \to \hat{M}$ is a submersion such that $\mathrm{Ker}\left((\pi_*)_p\right) = \mathrm{span}(R_p)$ and $\pi_*(\Xi_p) = T_{[p]}\hat{M}$ at any $p \in M$. Furthermore, one has a canonical lifting process $\lambda_p : T_{[p]}\hat{M} \to T_p M$ for any $p \in \pi^{-1}([p])$ by defining $\lambda_p(\hat{X}_{[p]})$ to be the unique vector in Ξ_p satisfying $\pi_*(\lambda_p\left(\hat{X}_{[p]}\right)) = \hat{X}_{[p]}$. As a result, whenever a structure on M is given that is equivariant with respect to the Reeb fibration, one obtains a similar structure on $\hat{M}$. For instance, $\hat{M}$ is a symplectic manifold because one can show that $\mathrm{d}\alpha$ is equivariant and the prescription $\hat{\omega}(\hat{X}, \hat{Y}) := \mathrm{d}\alpha\left(\lambda_p(\hat{X}), \lambda_p(\hat{Y})\right)$ for any pair of vector $\hat{X}, \hat{Y} \in T_{[p]}\hat{M}$ yields a well-defined symplectic form satisfying $\pi^*\hat{\omega} = \mathrm{d}\alpha$.

The diffeomorphisms of M that preserve the contact structure are called *contactomorphisms*. A contactomorphism ϕ must preserve the kernel of α, so that $\phi^*\alpha = \mathrm{e}^F \alpha$ for some function $F : M \to \mathbb{R}$. Thus if a one-parameter group of contactomorphisms of M is generated by a vector field X, then Lie differentiation shows that X must satisfy $\mathrm{d}(X \lrcorner \, \alpha) + X \lrcorner \, \mathrm{d}\alpha = \dot{F}\alpha$. One deduces that there is a function $u : M \to \mathbb{R}$ so that $X := X_u$ satisfies

$$\alpha(X_u) = u \qquad \text{and} \qquad X_u \lrcorner \, \mathrm{d}\alpha\big|_\Xi = -\mathrm{d}u. \qquad (1)$$

The vector field X_u is the *contact Hamiltonian vector field* generated by u. Note that if u is Reeb-equivariant, then u descends to a function $\hat{u} : \hat{M} \to \mathbb{R}$ and $\pi_*(X_u)$ is the Hamiltonian vector field of $\hat{M}$ generated by the function $\hat{u}$. Conversely, Hamiltonian vector fields of $\hat{M}$ can be lifted to equivariant contact-Hamiltonian vector fields of M. Moreover, if ϕ^t is the one-parameter group of contactomorphisms generated by u and $\hat{\phi}^t$ is the one-parameter group of symplectomorphisms generated by $\hat{u}$, then it is straightforward to check that ϕ^t is Reeb-equivariant and $\pi \circ \phi^t(p) = \hat{\phi}^t([p])$ for all $p \in [p]$. When the circumstance described here holds, one says that ϕ^t *covers* $\hat{\phi}^t$.

Suppose that $\Lambda \subseteq M$ is an integral submanifold of Ξ (i.e. $T_p\Lambda$ is a subspace of Ξ_p for every $p \in \Lambda$), then the full non-integrability condition implies that $\mathrm{d}\alpha\big|_\Lambda = 0$. But since $\mathrm{d}\alpha$ is a symplectic form on each contact hyperplane, the tangent spaces of Λ are isotropic subspaces of the contact hyperplanes and hence their dimension can be no larger that n. An integral submanifold of Ξ with this maximal dimension is called a *Legendrian* submanifold. It is easy to see that any Legendrian submanifold of M projects under π to a Lagrangian submanifold of $\hat{M}$. Conversely, if L is any Lagrangian submanifold of $\hat{M}$, then each of its tangent spaces can be lifted to a family of subspaces of Ξ. If $\hat{X}$ and $\hat{Y}$ are two tangent vectors of L, then the calculation $\alpha\left([\lambda(\hat{X}), \lambda(\hat{Y})]\right) = -\mathrm{d}\alpha\left(\lambda(\hat{X}), \lambda(\hat{Y})\right) = \hat{\omega}(\hat{X}, \hat{Y}) = 0$ shows that the sub-distribution of Ξ formed in this way is integrable in the sense of the Frobenius theorem. The integral submanifolds of this sub-distribution are a family of Legendrian submanifolds of M projecting onto L under π.

There is a great abundance of Legendrian submanifolds in $\mathbb{R}^{2n+1}$ with the contact structure α which are graphs of functions. The following proposition describes these fully.

Proposition 4 *Let* $\Pi = \mathbb{R}^n \times \{0\} \times \{0\} \subseteq \mathbb{R}^{2n} \times \mathbb{R}$ *with Lagrangian projection* $[\Pi] = \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{2n}$.

1. *Suppose* L *is a Lagrangian submanifold of* $\mathbb{R}^{2n}$ *that is graphical over* $[\Pi]$. *Then there exists a function* $f : \mathbb{R}^n \to \mathbb{R}$ *so that* $L = L_f$ *where* $L_f := \{(x, \nabla f(x)) : x \in \mathbb{R}^n\}$.
2. *Furthermore,* L *can be lifted to the family of graphical Legendrian submanifolds* $\Lambda_{f,c}$ *in* $\mathbb{R}^{2n} \times \mathbb{R}$ *where* $\Lambda_{f,c} := \{(x, \nabla f(x), 2f(x) + \sum_{k=1}^n x^k \frac{\partial f}{\partial x^k} + c) : x \in \mathbb{R}^n\}$.
3. *If* Λ *is a Legendrian submanifold in* $\mathbb{R}^{2n} \times \mathbb{R}$ *that is graphical over* Π *then there exists* $c \in \mathbb{R}$ *and* $f : \mathbb{R}^n \to \mathbb{R}$ *so that* $\Lambda = \Lambda_{f,c}$.

A consequence of this result is that Gray's Theorem now implies that Legendrian submanifolds of an arbitrary contact manifold M also come in great abundance, at least locally.

A final result about Legendrian submanifolds is called the *Legendrian Neighbourhood Theorem* and it asserts that there is a contactomorphism between a tubular neighbourhood of any Legendrian submanifold $\Lambda \subseteq M$ and a tubular neighbourhood of the zero section in $T^*\Lambda \times \mathbb{R}$ endowed with its canonical contact form $dt - \sigma$, where σ is the canonical one-form of $T^*\Lambda$. Thus any nearby Legendrian submanifold Λ' that is graphical over Λ, when pulled back under this contactomorphism, is the one-jet of a function on Λ. One can conclude from this the important fact that *Legendrian deformations* of Λ (i.e. deformations of Λ through Legendrian submanifolds via a one-parameter family of contactomorphisms) are generated by functions on Λ.

Sasakian geometry An odd-dimensional manifold M equipped with both a contact structure and a Riemannian metric, compatible with each other in the nicest way, is known as a *Sasakian structure* on M. A manifold with a Sasakian structure is the most general possible arena in which to envisage the equations studied in this paper. There are several ways of defining a Sasakian structure, the most germane of which is essentially taken from [2], with a few modifications and additions to suit the needs of this paper.

Definition 5 *Suppose* M *is an odd-dimensional manifold that carries a Riemannian metric* g *and a Killing field* R *of constant length 2. Define the endomorphism* $J \in End(TM)$ *by* $J(X) = \frac{1}{2}\nabla_X R$ *(so that* $J(R) = 0$*). Then* (g, R, J) *is a* Sasakian structure *for* M *if the following condition is met:*

$$(\nabla_X J)(Y) = \frac{1}{2}[g(R, Y)X - g(X, Y)R]$$

for all vector fields X, Y *on* M. *If the metric of* M *is Einstein and satisfies* $\mathrm{Ric}_g = 2ng$ *then* M *is said to be* Sasaki–Einstein.

Immediate consequences of the definition are the following.

Theorem 6 *Suppose* M *carries the Sasakian structure* (g, R, J) *and let* $\alpha = g(R, \cdot)$ *be the metric dual one-form of* R.

1. *The one-form* α *is a contact form whose contact structure* $\Xi := \mathrm{Ker}(\alpha)$ *is orthogonal to* R *and has Reeb vector field equal to* R.
2. *The characteristic foliation of* M *consists of geodesics and the metric* g *is Reeb-equivariant.*

3. *The endomorphism J is skew-symmetric, Reeb-equivariant, preserves Ξ, and satisfies*

$$d\alpha(X, Y) = g(JX, Y)$$

for all vector fields X, Y on M. Furthermore, $J^2 = -I + R \otimes \alpha$ where I is the identity.
4. *The Riemann curvature of g satisfies $\mathrm{Riem}_g(X, R)Y = g(X, Y)R - g(R, Y)X$ for all vector fields X, Y on M.*

From now on, suppose that the characteristic foliation of M consists of smooth, one-dimensional, embedded submanifolds; in this case the Sasakian structure of M is said to be *regular* and the fiber space $\hat{M}$ is once again a smooth manifold of dimension $2n$. The canonical lifting process described earlier induces two additional structures on $\hat{M}$ that come from equivariant structures of M, namely a metric $\hat{g}$ defined by $\hat{g}(\hat{X}, \hat{Y}) := g(\lambda(\hat{X}), \lambda(\hat{Y}))$ as well as an endomorphism $\hat{J}$ of the tangent spaces of $\hat{M}$ defined by $\hat{J}(\hat{X}) = \pi_* \circ J \circ \lambda(\hat{X})$.

Theorem 7 *Suppose (M, g, R, J) is a Sasakian manifold and let $\hat{M}$ be the fiber space of its characteristic foliation. Let $(\hat{g}, \hat{\omega}, \hat{J})$ be the structures on $\hat{M}$ defined above. Then the following hold.*

1. *$(\hat{M}, \hat{g}, \hat{\omega}, \hat{J})$ is a Kähler manifold of real dimension $2n$.*
2. *If M is Sasaki–Einstein, then $\hat{M}$ is Kähler–Einstein with Ricci curvature $\mathrm{Ric}_{\hat{g}} = (2n+2)\hat{g}$.*

If a contact manifold M possesses a Sasakian structure, then one can ask how the intrinsic and extrinsic geometry of its Legendrian submanifolds compares with that of their Lagrangian images under projection to the fiber space $\hat{M}$. The next theorem uses classical facts about Riemannian submersions [23] to answer this question. Denote by $B_{V,W}$ the second fundamental form of a submanifold V in an ambient manifold W.

Theorem 8 *Suppose M is a Sasakian manifold and $\hat{M}$ is its fiber space. If Λ is a Legendrian submanifold of M that projects onto a Lagrangian submanifold $[\Lambda] \subseteq \hat{M}$ under π, then the following are true.*

1. *The projection $\pi\big|_\Lambda : \Lambda \to [\Lambda]$ is a local isometry when Λ and $[\Lambda]$ are given their induced metrics in M and $\hat{M}$, respectively.*
2. *The second fundamental forms of Λ and $[\Lambda]$ satisfy*

$$B_{\Lambda, M}(X, Y) = \lambda \left(B_{[\Lambda], \hat{M}}(\pi_*(X), \pi_*(Y)) \right) \quad and$$

$$\pi_* \left(B_{\Lambda, M}(\lambda(\hat{X}), \lambda(\hat{Y})) \right) = B_{[\Lambda], \hat{M}}(\hat{X}, \hat{Y})$$

for all X, Y tangent to Λ and $\hat{X}$, $\hat{Y}$ tangent to $[\Lambda]$.

An important conclusion to be drawn from this theorem is that any minimal and Legendrian submanifold of M projects isometrically to a Lagrangian submanifold of $\hat{M}$ that is minimal; and any minimal and Lagrangian submanifold of $\hat{M}$ lifts isometrically to a family of Legendrian submanifolds of M that are all minimal.

Contact stationary Legendrian submanifolds   In a Sasakian manifold, one can consider the variation of the volume of a Legendrian submanifold Λ with respect to a restricted class of variations, namely the contact deformations of Λ. The Euler–Lagrange equation of this variational problem is given in the following theorem, in essence proved by Schoen and Wolfson [25].

Theorem 9 *Let Λ be a Legendrian submanifold of a Sasakian manifold M. If the volume of Λ is stationary with respect to all contact deformations of Λ, then $\nabla \cdot \left(H_\Lambda \lrcorner \, d\alpha \big|_\Lambda \right) = 0$, where H_Λ is the mean curvature of Λ.*

Proof By Theorem 8 and the fact that contact deformations of a Legendrian submanifold Λ of M cover symplectic deformations of $[\Lambda]$ in $\hat{M}$, it is sufficient to perform the calculation in the fiber space $\hat{M}$. Without loss of generality, let $\hat{\phi}_u$ be a symplectomorphism generated by a function $u : [\Lambda] \to \mathbb{R}$ with Hamiltonian vector field $X_u = \hat{J}\hat{\nabla}\hat{u}$ where $\hat{u}$ is a normal extension of u to a neighbourhood of $[\Lambda]$, and let $[\Lambda_t] = \phi_{tu}([\Lambda])$. Then stationarity implies

$$0 = \frac{d}{dt} Vol\left([\Lambda_t]\right)\Big|_{t=0} = -\int_{[\Lambda]} \hat{g}(H_{[\Lambda]}, \hat{J}\hat{\nabla}\hat{u}) \, d\text{Vol}_{\hat{g}} = -\int_{[\Lambda]} u \, \hat{\nabla} \cdot \left(H_{[\Lambda]} \lrcorner \, \hat{\omega} \big|_{[\Lambda]} \right) \, d\text{Vol}_{\hat{g}} \quad (2)$$

by Stokes' Theorem. Lifting to M, this is now just equivalent to $\int_\Lambda u \, \nabla\left(H_\Lambda \lrcorner \, d\alpha \big|_\Lambda \right) d\text{Vol}_g = 0$ for all u, from which the result follows. $\qquad\square$

Definition 10 *Let Λ be a Legendrian submanifold of a Sasakian manifold M. Then Λ is called* contact-stationary *if $\nabla \cdot \left(H_\Lambda \lrcorner \, d\alpha \big|_\Lambda \right) = 0$.*

In a Sasakian manifold, one has the notion of a *normal deformation* of a Legendrian submanifold Λ; that is, the infinitesimal vector field of the deformation is everywhere normal to Λ. It will be shown that given a sufficiently small function (in a C^1 norm) $u : \Lambda \to \mathbb{R}$, there is a one-parameter family of normal contactomorphisms with infinitesimal deformation field $J\nabla u$ on Λ. Denote the time-one contactomorphism simply by ϕ_u. Since it depends on the first derivatives of u and finding the divergence of the mean curvature of $\phi_u(\Lambda)$ takes another three derivatives, the operator

$$u \mapsto \nabla \cdot \left(H_{\phi_u(\Lambda)} \lrcorner \, d\alpha \Big|_{\phi_u(\Lambda)} \right) \quad (3)$$

is a non-linear, fourth-order, partial differential operator. The following terminology for this operator will be used from now on.

Definition 11 *The* contact-stationary Legendrian deformation operator *of the Legendrian submanifold Λ is defined to be the operator $\Phi_\Lambda : C^{4,\beta}(\Lambda) \to C^{0,\beta}(\Lambda)$ given in Eq. (3).*

Definition 12 *The* linearization *of Φ_Λ at f is the linear operator defined by the prescription $u \mapsto \frac{d}{dt}\Phi_\Lambda(f + tu)\big|_{t=0}$. It will be denoted $D\Phi_\Lambda(f)(u)$ or $\mathcal{L}_\Lambda$ if f is clear from the context.*

This linear operator will now be calculated at $f = 0$ in the general context of Sasakian geometry. Einstein summation convention is used where necessary below: i.e. repeated indices are summed, a comma denotes partial differentiation, a semi-colon denotes covariant differentiation with respect to the induced metric of Λ, indices are lowered and raised with this metric and its inverse, respectively, and so on. The calculation proceeds in two steps.

Proposition 13 *Let Λ be a Legendrian submanifold of a Sasaki–Einstein manifold M and suppose $\phi_u : M \to \mathbb{R}$ is a normal contact deformation generated by a function $u : \Lambda \to \mathbb{R}$. The infinitesimal variation of the mean curvature tensor of Λ is given by*

$$\frac{d}{dt}\Big|_{t=0} \left(H_{\Lambda_t} \lrcorner \, d\alpha \Big|_{\Lambda_t} \right) = d\left(\Delta_\Lambda u + (2n + 2)u \right) \quad (4)$$

where $\Lambda_t = \phi_{tu}(\Lambda)$ and Δ_Λ is the Laplace operator of Λ.

Proof According to Theorem 8, it is sufficient to calculate the infinitesimal variation of the mean curvature of the projected Lagrangian submanifold $[\Lambda]$ in the fiber space $\hat{M}$. Moreover, it is easy to see that there is a one-parameter family of symplectomorphisms $\hat{\phi}^t : \hat{M} \to \hat{M}$ such that ϕ_{tu} covers $\hat{\phi}^t$ in a tubular neighbourhood of $[\Lambda]$, and that the Hamiltonian vector field $\hat{X}$ is normal to $[\Lambda]$ with $\hat{X} = \hat{J}\nabla u$ along $[\Lambda]$, where ∇ is the induced connection of $[\Lambda]$.

One begins by identifying a good local frame with which to perform the calculations. Let $E_1, \ldots, E_n$ be a Riemannian normal coordinate frame for $[\Lambda]$ centered at a point $[p] \in [\Lambda]$. Extend these vectors by parallelism to a neighbourhood so that the vectors $E_1, \ldots, E_n, \hat{J}E_1, \ldots, \hat{J}E_n$ span the tangent spaces of $\hat{M}$ there. In this frame, the components of the second fundamental form $B_{[\Lambda]}$ and mean curvature $H_{[\Lambda]}$ are

$$B_{ijk} = \hat{g}(\hat{\nabla}_{E_i} E_j, JE_k) \quad \text{and} \quad H_k = h^{ij}\hat{g}(\hat{\nabla}_{E_i} E_j, JE_k)$$

where $\hat{\nabla}$ is the ambient connection of $\hat{M}$ and h^{kl} are the components of the inverse of the induced metric h of $[\Lambda]$, whose components are $h_{kl} = \hat{g}(E_k, E_l) = \hat{g}(\hat{J}E_k, \hat{J}E_l)$ because $\hat{J}$ is an isometry. Note that B_{ijk} is symmetric in all its indices.

The time derivative of the components H_k will be computed at the point $[p]$ itself, where the Christoffel symbols of h vanish, partial differentiation and covariant differentiation coincide, and therefore $\hat{\nabla}_{E_i} E_j = B_{ij}^k \hat{J}E_k$. Recall that it is possible to assume $\hat{\nabla}_X E_i = \hat{\nabla}_{E_i} X$, so that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} H_k\Big|_{t=0} &= X(h^{ij})\hat{g}(\hat{\nabla}_{E_i} E_j, \hat{J}E_k) + h^{ij} X\hat{g}(\nabla_{E_i} E_j, \hat{J}E_k) \\
&= 2B_{ijk}B^{ijl}u_{;l} + h^{ij}\left(\hat{g}(\hat{\nabla}_X \hat{\nabla}_{E_i} E_j, JE_k) + \hat{g}(\hat{\nabla}_{E_i} E_j, J\hat{\nabla}_X E_k)\right) \\
&= h^{ij}\left(\hat{g}(\hat{\nabla}_{E_i} \hat{\nabla}_{E_j} X, JE_k) + u^{;l}\,\mathrm{Riem}_{\hat{g}}(E_i, JE_l, E_j, JE_k)\right) \\
&\quad - H^l B_{lk}^m u_{,m} + 2B_{ijk}B^{ijl}u_{;l}
\end{aligned} \tag{5}$$

since $\hat{\nabla}_{E_k} X = \hat{J}\hat{\nabla}_{E_k}\nabla u = u^l_{;k}\hat{J}E_l - u^{;l}B_{kl}^m E_m$. Here $\mathrm{Riem}_{\hat{g}}$ is the Riemannian curvature tensor of $\hat{M}$. Calculate the remaining term containing X as follows:

$$\begin{aligned}
\hat{g}(\hat{\nabla}_{E_i} \hat{\nabla}_{E_j} X, JE_k) &= \hat{g}(\hat{\nabla}_{E_i} \hat{\nabla}_{E_j} \nabla u, E_k) \\
&= E_i \hat{g}(\nabla_{E_j}\nabla u + B_{[\Lambda]}(E_j, \nabla u), E_k) - \hat{g}(\nabla u + B_{[\Lambda]}(E_j, \nabla u), \nabla_{E_j} E_k) \\
&= h(\nabla_{E_i}\nabla_{E_j}\nabla u, E_k) - \hat{g}(B_{[\Lambda]}(E_j, \nabla u), B_{[\Lambda]}(E_i, E_k)) \\
&= u_{;kij} - B_j^{lm} B_{ikl} u_{;m}
\end{aligned} \tag{6}$$

Finally, perform the following manipulation on the $\mathrm{Riem}_{\hat{g}}$ term. Start by adding and subtracting exactly the quantity required to yield the full Ricci curvature of $\hat{M}$. That is,

$$\begin{aligned}
h^{ij}\mathrm{Riem}_{\hat{g}}(E_i, \hat{J}E_l, E_j, \hat{J}E_k) &= h^{ij}\mathrm{Riem}_{\hat{g}}(E_i, \hat{J}E_l, E_j, \hat{J}E_k) \\
&\quad + h^{ij}\mathrm{Riem}_{\hat{g}}(\hat{J}E_i, \hat{J}E_l, \hat{J}E_j, \hat{J}E_k) \\
&\quad - h^{ij}\mathrm{Riem}_{\hat{g}}(\hat{J}E_i, \hat{J}E_l, \hat{J}E_j, \hat{J}E_k) \\
&= (2n+2)h_{kl} - h^{ij}\mathrm{Riem}_{\hat{g}}(E_i, E_l, E_j, E_k).
\end{aligned} \tag{7}$$

Complete the calculation by applying the Gauß equation, which reads

$$h^{ij}\mathrm{Riem}_{\hat{g}}(E_i, E_l, E_j, E_k) = R_{kl} - H^m B_{mkl} + B_k^{mn} B_{lmn}, \tag{8}$$

where R_{kl} are the components of the Ricci tensor of Λ. Substituting (6), (7) and (8) into (5) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} H_k \Big|_{t=0} = \Delta_\Lambda(u_{;k}) - u_{;l} R_k^l - (2n+2)u_{;k} = (\Delta_\Lambda u)_{;k} - (2n+2)u_{;k}.$$

This is the desired formula. $\qquad\square$

Corollary 14 *Let Λ be a Legendrian submanifold of a Sasaki-Einstein manifold M and suppose $\phi_u : M \to \mathbb{R}$ is a normal contact deformation generated by a function $u : \Lambda \to \mathbb{R}$. The infinitesimal variation of the divergence of the mean curvature tensor of Λ is given by*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \nabla \cdot \left(H_{\Lambda_t} \lrcorner \, \mathrm{d}\alpha \Big|_{\Lambda^t} \right) = \Delta_\Lambda \left(\Delta_\Lambda u + (2n+2)u \right) + Q_\Lambda(u) \tag{9}$$

where $\Lambda_t = \phi_{tu}(\Lambda)$ and Q_Λ is the operator given by $Q_\Lambda(u) = 2\,(\nabla \cdot B_\Lambda)(H_\Lambda, \nabla u) - (H_\Lambda \cdot \nabla)^2 u$.

Proof Compute the time derivative of $h^{ij} H_{i;j} = h^{ij}(H_{i,j} - H_s \Gamma_{ij}^s)$ in terms of the derivatives of h^{ij} and Γ_{ij}^s in the same way as above. $\qquad\square$

A standard fact about the linearization of the contact-stationary Legendrian operator is that one-parameter families of isometries that are also contactomorphisms (which will be called *contact isometries*) produce elements in the kernel of its linearization.

Corollary 15 *Suppose $\phi_f^t : M \to M$ is a one-parameter family of contact isometries of M generated by a function $f : M \to \mathbb{R}$. If Λ is a Legendrian submanifold of M then $\mathrm{D}\Phi_\Lambda(0)(f|_\Lambda) = 0$.*

Proof This follows from the fact that $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \nabla \cdot \left(H_{\phi_f^t(\Lambda)} \lrcorner \, \mathrm{d}\alpha|_{\phi_f^t(\Lambda)} \right) = 0$. $\qquad\square$

2.2 The geometry of the unit sphere

Geometric structures The Calabi–Yau structure of $\mathbb{C}^{n+1}$ consists of the standard Euclidean metric δ, the standard symplectic form ω_0 and the standard complex structure J_0 satisfying $\omega_0(X, Y) = \delta(J_0 X, Y)$, as well as the canonical holomorphic volume form $\Omega := \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^{n+1}$. The Calabi–Yau structure of $\mathbb{C}^{n+1}$ induces a Sasaki–Einstein structure on $\mathbb{S}^{2n+1}$. First, denote the position vector field by

$$P := \sum_{k=1}^{n+1} \left(x^k \frac{\partial}{\partial x^k} + y^k \frac{\partial}{\partial y^k} \right) = \sum_{k=1}^{n+1} \left(z^k \frac{\partial}{\partial z^k} + \bar{z}^k \frac{\partial}{\partial \bar{z}^k} \right),$$

given both in real and complex coordinates. Then the relevant objects are the following.

- The metric is the standard metric of the sphere, induced from the ambient Euclidean metric.
- The contact form is

$$\alpha := \frac{1}{2} P \lrcorner \, \omega_0 \big|_{\mathbb{S}^{2n+1}} = \frac{1}{2} \sum_{k=1}^{n+1} \left(x^k \mathrm{d}y^k - y^k \mathrm{d}x^k \right) \Big|_{\mathbb{S}^{2n+1}} = \frac{1}{4\mathrm{i}} \sum_{k=1}^{n+1} \left(\bar{z}^k \mathrm{d}z^k - z^k \mathrm{d}\bar{z}^k \right) \Big|_{\mathbb{S}^{2n+1}}$$

so that $\mathrm{d}\alpha = \omega_0 \big|_{\mathbb{S}^{2n+1}}$.

- The Reeb vector field is $R := 2J_0P$ so that the contact structure is $\Xi_p := \left(\mathrm{span}(J_0P_p)\right)^\perp$ for every $p \in \mathbb{S}^{2n+1}$.
- The endomorphism J is defined to equal J_0 on Ξ and to vanish in the Reeb direction.

The characteristic foliation of $\mathbb{S}^{2n+1}$ as a Sasakian manifold coincides with the *Hopf fibration* and the fiber space of $\mathbb{S}^{2n+1}$ coincides with the Kähler manifold $\mathbb{CP}^n$. Furthermore, the Reeb projection coincides with the Hopf projection $\pi_H : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ which is the Riemannian submersion of the sphere onto $\mathbb{CP}^n$ with the Fubini-Study metric. To see all this, recall that $\mathbb{CP}^n$ is the space of complex lines in $\mathbb{C}^{n+1}$. Thus $\mathbb{CP}^n$ is the orbit space of the action of multiplication by non-zero complex numbers restricted to the sphere, i.e. the $\mathbb{S}^1$ action $\theta \cdot p := e^{i\theta} p$ for $p \in \mathbb{S}^{2n+1}$. But the differential of this action is $\frac{\mathrm{d}}{\mathrm{d}\theta}(\theta \cdot p)|_{\theta=0} = ip$ which is exactly the value of the vector field JP at p (in complex coordinates). Hence the orbits of the action coincide with the characteristic foliation.

Contact isometries of the sphere. The isometries of $\mathbb{S}^{2n+1}$ that preserve the Sasakian structure derive from the complex structure-preserving isometries of Euclidean space, namely the $U(n+1)$-rotations. The one-parameter subgroups of $U(n+1)$ will play a crucial role in the sequel. If U^t is such a subgroup with $U^0 = Id$, then there is a Hermitian matrix H so that $U^t = \exp(iHt)$. Also, U^t is a one-parameter family of symplectomorphisms of $\mathbb{C}^{n+1}$ whose Hamiltonian vector field is given by $X_H(z) := iHz$ for any $z \in \mathbb{C}^{n+1}$. The Hamiltonian function associated to X_H is the Hermitian, harmonic, homogeneous polynomial of degree 2 given by $q_H(z) := z^*Hz$. Moreover, U^t restricts to a contactomorphism of $\mathbb{S}^{2n+1}$ with contact vector field X_H and contact Hamiltonian equal to the restriction $q_H|_{\mathbb{S}^{2n+1}}$. Finally, since $U(n+1)$-rotations are contact isometries of $\mathbb{S}^{2n+1}$, Corollary 15 implies that $\mathcal{L}_\Lambda \left(q_H|_\Lambda\right) = 0$ where $\mathcal{L}_\Lambda$ is the linearization of the contact-stationary Legendrian deformation operator. Hence the kernel of this linearized operator is always non-trivial.

The following explicit specification of the generators of $U(n+1)$ and their associated Hermitian, harmonic, homogeneous polynomials of degree 2 on $\mathbb{C}^{n+1}$ will be needed in the sequel. A basis for the set of complex $(n+1) \times (n+1)$ Hermitian matrices is given by:

$$I := \begin{pmatrix} 1 \\ & \ddots \\ & & 1 \end{pmatrix} \quad H_1 := \begin{pmatrix} n \\ & -1 \\ & & \ddots \\ & & & -1 \end{pmatrix} \quad H_2 := \begin{pmatrix} 0 \\ & 1 \\ & & 0 \\ & & & \ddots \\ & & & & -1 \end{pmatrix} \quad \cdots \quad H_n := \begin{pmatrix} 0 \\ & 0 \\ & & \ddots \\ & & & 1 \\ & & & & -1 \end{pmatrix},$$

which are the real, diagonal matrices; along with

$$H_{jk} := \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \qquad H'_{jk} := \begin{pmatrix} & i \\ -i & \end{pmatrix},$$

which are the symmetric matrices having 1 in the jth row and kth column, and the anti-symmetric matrices having $i = \sqrt{-1}$ in the jth row and kth column. Their associated polynomials are:

$$q_0(z) := z^* I z = \sum_{s=0}^{n} |z^s|^2$$

$$q_1(z) := z^* H_1 z = n|z^1|^2 - \sum_{s=2}^{n+1} |z^s|^2$$

$$q_j(z) := z^* H_j z = |z^j|^2 - |z^{n+1}|^2 \quad \text{for } j = 2, \ldots, n \tag{10}$$

$$q_{\text{RE}, jk}(z) := z^* H_{jk} z = 2 \, \mathbf{Re}(z^j \bar{z}^k) \quad \text{for } 1 \leq j < k \leq n+1$$

$$q_{\text{IM}, jk}(z) := z^* H'_{jk} z = 2 \, \mathbf{Im}(z^j \bar{z}^k) \quad \text{for } 1 \leq j < k \leq n+1.$$

3 The Legendrian connected sum procedure

The present section of the paper describes the *Legendrian connected sum procedure* that will be used to connect a chain of minimal Legendrian $U(n+1)$-rotated n-spheres together. For the moment, consider a single n-sphere S_0 and two neighbouring n-spheres $S_1 := U_1(S_0)$ and $S_2 := U_2(S_0)$ with the following properties. There is a small $c \in (0, 2\pi)$ so that S_0 and S_j have Hopf separation c at the point p_j; and the tangent spaces $T_{p_j} S_0$ and $T_{p_j} e^{(-1)^j \mathrm{i} c} S_j$ are transverse and satisfy the angle criterion. The points p_1 and p_2 are the *gluing points* on S_0. Note that the Hopf separation between S_0 and S_1 is equal but opposite to the Hopf separation between S_0 and S_2. Without loss of generality $S_0 := \mathbb{R}^{n+1} \times \{0\} \cap \mathbb{S}^{2n+1}$.

3.1 Preliminary perturbation of the n-spheres

In order to connect S_0 to S_1 and S_2 at the gluing points with the best possible estimate of the mean curvature of the connected sum, it is first necessary to deform these n-sphere in a canonical manner. The point is to 'prepare' them for the connected sum operations by giving them a catenoidal shape near the points of connection. To this end, let ε be a small, positive number (which will depend on c in a manner to be determined below) and choose the unique distributional solution $G_0 : S_0 \setminus \{p_1, p_2\} \to \mathbb{R}$ of the equation $\Delta_S(G_0) = \delta_1 - \delta_2$ satisfying $\int_{S_0} G_0 = 0$, where δ_j is the Dirac δ-mass at p_j and Δ_S is the Laplacian of the induced metric of S_0. Choose similar functions for each S_j, but using the gluing points where S_j is to be connected to its two neighbours (one of which is S_0). Henceforth consider only S_0 and the function G_0 since the analysis on the neighbouring n-spheres is the same. Note that

$$G_0(x) = \begin{cases} \dfrac{(-1)^{j+1}}{[\text{dist}(x, p_j)]^{n-2}} + O(1) & n \geq 3 \\[2mm] (-1)^j \log[\text{dist}(x, p_j)] + O(1) & n = 2 \end{cases} \tag{11}$$

in a neighbourhood of each p_j.

The idea is now to replace each S_0 by a Legendrian perturbation of S_0 generated by $\varepsilon^n G_0$. The way to do this is fairly simple and depends on the fact that the intersection of a Lagrangian cone in $\mathbb{C}^{n+1}$ with $\mathbb{S}^{2n+1}$ is a Legendrian submanifold. Let $\bar{G}_0 : \mathbb{R}^{n+1} \times \{0\} \to \mathbb{R}$ denote the degree-two homogeneous extension of G_0 to the $(n+1)$-plane containing S_0, namely the function defined by $\bar{G}_0(x) := \|x\|^2 G_0(x/\|x\|)$ for $x \in \mathbb{R}^{n+1} \times \{0\}$. Therefore the submanifold

$$n\overline{\text{Pert}_\varepsilon(S_0)} := \left\{ (x, \varepsilon^n \mathring{\nabla} \bar{G}_0(x)) : x \in \mathbb{R}^{n+1} \right\},$$

where $\mathring{\nabla}$ denotes the Euclidean gradient, is Lagrangian and a cone; and thus intersecting with $\mathbb{S}^{2n+1}$ yields a Legendrian submanifold.

Definition 16 *The* preliminary perturbation *of* S_0 *with parameter* ε *is the Legendrian submanifold* $\mathrm{Pert}_\varepsilon(S_0) := \overline{\mathrm{Pert}_\varepsilon(S_0)} \cap \mathbb{S}^{2n+1}$.

The induced metric and second fundamental form of S_0 are trivial (i.e. the standard metric and zero, respectively) and this changes when S_0 is replaced by $\mathrm{Pert}_\varepsilon(S_0)$. But if ε is sufficiently small, then the geometry of $\mathrm{Pert}_\varepsilon(S_0)$ changes only slightly, and this is true even relatively near to the gluing points. This will be quantified in the following proposition. The key is to exploit the relationship between a geometric object on $\mathrm{Pert}_\varepsilon(S_0)$ and the corresponding geometric object on $\overline{\mathrm{Pert}_\varepsilon(S_0)}$. Also $\|\cdot\|^*_{C^k(\mathcal{O})}$ denotes the so-called *scale-invariant* norm on a neighbourhood $\mathcal{O}$ in which the jth derivative is weighted by the factor r^j where $r := diam(\mathcal{O})$.

Proposition 17 *Let* r_0 *denote a radius so that* (11) *is valid inside* $B_r(p_j)$ *for all* $0 < r < r_0$ *and set* $S' := \mathrm{Pert}_\varepsilon(S_0) \setminus \left[B_{r_0}(p_1) \cup B_{r_0}(p_2) \right]$. *Then the following estimates for geometric objects on* $\mathrm{Pert}_\varepsilon(S_0)$ *hold.*

- *The second fundamental form satisfies the estimate*

$$\|B\|^*_{C^2(B_r(p_j))} \le \frac{C\varepsilon^n}{r^{n+1}} \quad \forall r \in (0, r_0) \quad and \quad \|B\|_{C^2(S')} \le C\varepsilon^n.$$

- *The mean curvature satisfies the estimate*

$$|\nabla \cdot H|^*_{C^1(B_r(p_j))} \le \frac{C\varepsilon^{3n}}{r^{3n+2}} \quad \forall r \in (0, r_0) \quad and \quad |\nabla \cdot H|_{C^1(S')} \le C\varepsilon^{3n}$$

$$\|H\|^*_{C^2(B_r(p_j))} \le C\frac{\varepsilon^n}{r^{n-1}} \quad and \quad \|H\|_{C^2(S')} \le C\varepsilon^n.$$

- *The Laplacian satisfies*

$$|\Delta(u) - \Delta_S(u)|^*_{C^1(B_r(p_j))} \le \frac{C\varepsilon^{2n}}{r^{2n+2}} |u|^*_{C^2(B_r(p_j))} \quad \forall r \in (0, r_0)$$

and

$$|\Delta(u) - \Delta_S(u)|_{C^1(S')} \le C\varepsilon^{2n} |u|_{C^2(S')}$$

where Δ_S *is the Laplacian of the standard n-sphere.*

Proof Let $\bar{h}, \bar{\nabla}, \bar{\Delta}, \bar{B}$ and $\bar{H}$ denote the induced metric, covariant derivative, Laplacian, second fundamental form and mean curvature of $\overline{\mathrm{Pert}_\varepsilon(S_0)}$ while h, ∇, Δ, B and H denote the same things for $\mathrm{Pert}_\varepsilon(S_0)$. For typographical convenience, set $\bar{G} := \varepsilon^n \bar{G}_0$ and $G := \varepsilon^n G_0$.

Simple computations show that on $\overline{\mathrm{Pert}_\varepsilon(S_0)}$ the induced metric is $\bar{h}_{ij} := \delta_{ij} + \sum_s \bar{G}_{,si} \bar{G}_{,sj}$, the Christoffel symbols are $\bar{\Gamma}_{ijk} := \sum_s \bar{G}_{,ijs} \bar{G}_{,sk}$ and the components of the second fundamental form are $\bar{B}_{ijk} := \bar{G}_{,ijk}$. Thus the inverse of $\bar{h}_{ij}$ equals the identity plus $O(\|\bar{\nabla}^2 \bar{G}\|^2)$ terms so that

$$\|\bar{\nabla}^k \bar{B}\| = O(\|\bar{\nabla}^{3+k} \bar{G}\|),$$

all other terms being smaller. Also,

$$
\begin{aligned}
\bar{\nabla} \cdot \bar{H} &= \bar{h}^{kl} \bar{h}^{ij} (\bar{B}_{,ijk)_{;l}} \\
&= \left[\delta^{kl} + O(\|\bar{\nabla}^2 \bar{G}\|^2) \right] \left[\delta^{ij} + O(\|\bar{\nabla}^2 \bar{G}\|^2) \right] \left[\bar{G}_{,ijkl} - 2\bar{G}_{,sjk} \bar{\Gamma}^s_{il} - \bar{G}_{,ijs} \bar{\Gamma}^s_{kl} \right] \\
&= O(\|\bar{\nabla}^2 \bar{G}\|^2 \|\bar{\nabla}^4 \bar{G}\|) + O(\|\bar{\nabla}^3 \bar{G}\|^2 \|\bar{\nabla}^2 \bar{G}\|).
\end{aligned}
$$

This is because $\delta^{kl} \delta^{ij} \bar{G}_{,ijkl} = \mathring{\Delta}(\mathring{\Delta}\bar{G})$ where $\mathring{\Delta}$ is the Euclidean Laplacian, and the choice of G implies $\mathring{\Delta}(\mathring{\Delta}\bar{G}) = \frac{1}{\|x\|^2} \Delta(\Delta + 2(n+1))(G) = 0$. An estimate of $\|H\|$ follows in a similar way to the previous two estimates, and yields

$$
\|H\| = O(\|\bar{\nabla}\bar{G}\|).
$$

This is not quite as good as before because the leading term in the expansion of $\|H\|$ is the derivative of the Laplacian of the function $\bar{G}$, which is the derivative of $\bar{G}$. Finally,

$$
\begin{aligned}
\bar{\Delta}(u) - \mathring{\Delta}(u) &= \bar{h}^{ij} \left[u_{,ij} - u_{,s} \bar{\Gamma}^s_{ij} \right] \\
&= O(\|\bar{\nabla}^2 \bar{G}\|^2) \cdot \bar{\nabla}^2 u + O(\|\bar{\nabla}^3 \bar{G}\| \cdot \|\bar{\nabla}^2 \bar{G}\|) \cdot \bar{\nabla} u
\end{aligned}
$$

where $O(*) \cdot$ denotes a linear operation on tensors with coefficients bounded by $O(*)$.

Now using the homogeneity of $\bar{G}$ and the fact that $\overline{\mathrm{Pert}_\varepsilon(S_0)}$ is a cone over $\mathrm{Pert}_\varepsilon(S_0)$, one can replace $\|\bar{\nabla}^k \bar{G}\|$ by $\sum_{s=0}^k \|\nabla^s G\|$ to obtain expressions valid on $\mathrm{Pert}_\varepsilon(S_0)$. Substituting $G = \varepsilon^n G_0$ and using (11) then yields the desired result. Higher derivative estimates follow in the same way.   □

3.2 Normal coordinates for $\mathbb{S}^{2n+1}$

The next step is to introduce canonical coordinates for $\mathbb{S}^{2n+1}$ near the gluing points in which all the relevant structures have an extremely simple form. These coordinates will be called *Legendrian normal coordinates* and are defined by first constructing a 'standard' coordinate map and then transplanting it to a neighbourhood of a desired point p by means of an $SU(n+1)$-rotation.

Endow $\mathbb{C}^{n+1}$ with coordinates $z := (z^0, z')$ where $z' := (z^1, \ldots, z^n)$ and define the half-space $\mathcal{H} = \{z \in \mathbb{C}^{n+1} : \mathbf{Re}(z^0) \le 0\}$. Next, endow $\mathbb{C}^n$ with coordinates $(w^1, \ldots, w^n)$ and let $B_1(0) \subseteq \mathbb{C}^n$ be the unit ball. Define the map $K : B_1(0) \times (-\pi, \pi) \to \mathbb{S}^{2n+1} \backslash \mathcal{H}$ by $K(w, t) := \mathrm{e}^{\mathrm{i}t} (f(w), w)$, where $f : \mathbb{C}^n \to \mathbb{R}$ is given by $f(w) := (1 - \|w\|^2)^{1/2}$. Then K is invertible and the inverse is given by $K^{-1}(z) = \left(\mathrm{e}^{-\mathrm{i}\arg(z^0)} z', \arg(z^0) \right)$. Now let $p \in \mathbb{S}^{2n+1}$ be any point and suppose Π is any Legendrian n-plane in $T_p \mathbb{S}^{2n+1}$. Let $e_0, e_1, \ldots, e_n$ be the standard basis of $\mathbb{C}^{n+1}$ and choose $V_{p,\Pi} \in SU(n+1)$ taking e_0 to p and the real linear span of $e_1, \ldots, e_n$ to Π. Note that $V_{p,\Pi}$ is unique up to orthogonal transformations and has the property that the contact hyperplane at e_0 (which is the complex linear span of $e_1, \ldots, e_n$) is mapped to the contact hyperplane at p and the Reeb direction at e_0 (which is the real line spanned by $\mathrm{i}e_0$) is mapped to the Reeb direction at p (which is the real real line spanned by $\mathrm{i}p$). The transformation $V_{p,\Pi}$ allows the map K to be transplanted to the point p in a manner adapted to Π. Composing these maps leads to the desired coordinates.

Definition 18 *The* Legendrian normal coordinate chart *at* $p \in \mathbb{S}^{2n+1}$ *adapted to the Legendrian n-plane* $\Pi \subseteq T_p\mathbb{S}^{2n+1}$ *is the map*

$$\psi_{p,\Pi} : \mathbb{S}^{2n+1}\backslash V_{p,\Pi}(\mathcal{H}) \to B_1(0) \times (-\pi, \pi)$$

defined by $\psi_{p,\Pi} := K^{-1} \circ V_{p,\Pi}^{-1}.$

The properties of Legendrian normal coordinates that will be used in the remainder of this paper are gathered in the following propositions. Denote the set $\mathbb{S}^{2n+1}\backslash V_{p,\Pi}(\mathcal{H})$ by $\mathcal{H}_{p,\Pi}$. Endow $\mathbb{C}^n$ with the standard symplectic structure $\omega_0 = \frac{1}{2i} \sum d\bar{w}^k \wedge dw^k$ and endow $\mathbb{C}^n \times \mathbb{R}$ with the contact structure defined by the contact form $\alpha_0 = \frac{1}{2}dt + \sigma_0$ where $\sigma_0 = \frac{1}{4i} \sum (\bar{w}^k dw^k - w^k d\bar{w}^k)$. Note that $\omega_0 = d\sigma_0$. Finally, define the projection $\pi_0 : \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^n$ by $\pi_0(w, t) = w$, and for each $c \in \mathbb{R}$ define the injection $i_c : \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{R}$ by $i_c(w) = (w, c)$. The proof of the next proposition is just straightforward calculation and diagram chasing.

Proposition 19 *Suppose* $p \in \mathbb{S}^{2n+1}$ *is and* $\Pi \subseteq T_p\mathbb{S}^{2n+1}$ *is Legendrian n-plane. The Legendrian normal coordinate chart* $\psi_{p,\Pi} : \mathcal{H}_{p,\Pi} \to B_1(0) \times (-\pi, \pi)$ *at p has the following properties.*

1. *The map* $\psi_{p,\Pi} : (\mathbb{S}^{2n+1}, \alpha) \to (\mathbb{C}^n \times \mathbb{R}, \alpha_0)$ *is a contactomorphism.*
2. *If* Λ *is a Legendrian submanifold tangent to* $e^{ic}\Pi$ *at* $e^{ic}p$ *then* $\psi_{p,\Pi}(\Lambda)$ *is a Legendrian submanifold of* $\mathbb{C}^n \times \mathbb{R}$ *tangent to the plane* $\mathbb{R}^n \times \{c\}$ *at* $(0, c)$*. Moreover,* $\psi_{p,\Pi}$ *takes* $U(n + 1)$*-rotated Legendrian n-spheres passing through* $e^{ic}p$ *to Legendrian n-planes passing through* $(0, c)$*.*
3. *The map* $\hat{\psi}_{[p],[\Pi]} : (\mathbb{CP}^n, \hat{\omega}) \to (\mathbb{C}^n, \omega_0)$ *defined by* $\hat{\psi}_{[p],\Pi]} := \left(\pi_H \circ \psi_{p,\Pi}^{-1} \circ i_0\right)^{-1}$ *is a symplectomorphism and the following diagram commutes.*

$$
\begin{array}{ccc}
\mathcal{H}_{p,\Lambda} \subseteq (\mathbb{S}^{2n+1}, d\alpha) & \xrightarrow{\psi_{p,\Pi}} & (\mathbb{C}^n \times \mathbb{R}, d\alpha_0) \\
\downarrow{\scriptstyle \pi_H} & & \pi_0 \downarrow \uparrow i_0 \\
[\mathcal{H}_{p,\Pi}] \subseteq (\mathbb{CP}^n, \hat{\omega}) & \xrightarrow{\hat{\psi}_{[p],\Pi]}} & (\mathbb{C}^n, \omega_0)
\end{array}
\qquad (12)
$$

The indicated differential forms correspond under pull-back by the appropriate mappings.
4. *Let g be the induced metric of* $\mathbb{S}^{2n+1}$ *and* $\hat{g}$ *be the Fubini-Study metric of* $\mathbb{CP}^n$*; and let* $g_0 := \left(\psi_{p,\Lambda}^{-1}\right)^* g$ *and* $\hat{g}_0 := \left(\hat{\psi}_{[p],[\Lambda]}^{-1}\right)^* \hat{g}$ *be the corresponding pull-back metrics. Then*

$$\pi_0 : (\mathbb{C}^n \times \mathbb{R}, g_0) \to (\mathbb{C}^n, \hat{g}_0)$$

is a Riemannian submersion. Moreover, these metrics are given by

$$g_0 = dt^2 + \delta + df \otimes df + 2(dt \otimes \sigma_0 + \sigma_0 \otimes dt)$$
$$\hat{g}_0 = \delta + df \otimes df - 4\sigma_0 \otimes \sigma_0$$

where δ *is the Euclidean metric of* $\mathbb{C}^n$*.*

It will be necessary to estimate quite precisely the intrinsic and extrinsic geometry of a Legendrian submanifold Λ in $\mathbb{C}^n \times R$ with respect to the induced metric g_0. A consequence of Lemma 8 is that the necessary estimates can be gotten by estimating $[\Lambda] \subseteq \mathbb{C}^n$ with respect to the metric $\hat{g}_0$, which amounts to a significant simplification since $\hat{g}_0$ is much simpler than g_0.

A further simplification becomes available if the Lagrangian projection of the submanifold to be estimated is contained in a very small neighbourhood of the origin. This is because $\hat{g}_0$ is in Riemannian normal form at the origin (namely $(\hat{g}_0)_{ij} = \delta_{ij}$ and $(\hat{g}_0)_{ij,k} = 0$ at the origin, for all i, j, k) by virtue of the fact that $g = \delta + Q$ where the coefficients of the tensor Q satisfy $|Q_{ij}(z)| + \|z\| \, |Q_{ij,k}(z)| + \|z\|^2 \, |Q_{ij,kl}(z)| = O(\|z\|^2)$. Thus one would expect that the geometry of $[\Lambda]$ with respect to the metric $\hat{g}_0$ is uniformly close to the geometry of $[\Lambda]$ with respect to δ. The following proposition makes this idea rigorous. The following notation will be used: H^g, B^g, ∇^g, Δ^g, $\langle \cdot, \cdot \rangle_g$ and $\| \cdot \|_g$ denote the mean curvature, second fundamental form, covariant derivative, Laplacian, inner product and norm induced by a metric g.

Proposition 20 *There exists $r_0 > 0$ so that the following holds for all $0 < r < r_0$. Let $\hat{\lambda} : \mathcal{O} \to \mathbb{C}^n$ be a Lagrangian embedding of a neighbourhood $\mathcal{O} \subseteq \mathbb{R}^n$ such that $\hat{\lambda}(\mathcal{O}) \subseteq B_r(0)$ and put $h = \hat{\lambda}^* \hat{g}_0$.*

- *The mean curvature and second fundamental form of $\mathcal{O}$ satisfy the following estimates.*

$$\|B^{\hat{g}_0}\|^*_{C^2(\mathcal{O},h)} = \|B^\delta\|^*_{C^2(\mathcal{O},\hat{\lambda}^*\delta)} + O(r)$$

$$\|H^{\hat{g}_0}\|^*_{C^2(\mathcal{O},h)} = \|H^\delta\|^*_{C^2(\mathcal{O},\hat{\lambda}^*\delta)} + O(r) + O(r^2)\|B^\delta\|^*_{C^2(\mathcal{O},\hat{\lambda}^*\delta)}.$$

- *The Laplacian of $\mathcal{O}$ satisfies*

$$\Delta^h u = \Delta^\delta u + O(r) \cdot \nabla^\delta u + O(r^2) \cdot \nabla^\delta \nabla^\delta u.$$

where $O(*) \cdot$ denotes a linear operation on tensors with coefficients bounded by $O(*)$.

Proof The proof of this result is a rather diabolical exercise in Riemannian geometry that will be abbreviated here for the sake of the reader. First, pick any point $p \in \lambda(\mathcal{O})$ and choose a δ-orthonormal frame E_i for $T_p\lambda(\mathcal{O})$. Extend this to a δ-orthonormal frame for $T_p\mathbb{C}^n$ by adjoining the vectors JE_i. One can assume that $\langle \nabla^\delta_{E_i} E_j, E_k \rangle_\delta = 0$ and that $B^\delta_{ijk} = \langle \nabla^\delta_{E_i} E_j, JE_k \rangle_\delta$.

The second fundamental form of $\lambda(\mathcal{O})$ with respect to the metric h can be found by computing $B^h_{ijk} = \langle \nabla^h_{E_i} E_j, N^h_k \rangle_h$ where N^h_k is a h-orthonormal frame for the h-orthogonal complement of $T_p\lambda(\mathcal{O})$. To complete this calculation, one uses the form of the metric to compute $\langle \nabla^h_X Y, Z \rangle_h = \langle \nabla^\delta_X Y, Z \rangle_\delta + O(r)\|X\|_\delta \|Y\|_\delta \|Z\|_\delta$ for any vectors X, Y, Z whereas the N^h_k can be written as linear combinations of $JE_k + c^i_k E_i$ where $c^i_k = O(r^2)$. Substituting these quantities into the expression for B^h_{ijk} and its covariant derivatives yields

$$B^h_{ijk} = B^\delta_{ijk} + O(r)$$

$$\nabla^h_s B^h_{ijk} = \nabla^\delta_s B^\delta_{ijk} + O(1)$$

$$\nabla^h_s \nabla^h_t B^h_{ijk} = \nabla^\delta_s \nabla^\delta_t B^\delta_{ijk} + O(r^{-1})$$

where $O(r^*)$ can now represents a tensor of the required type having coefficients bounded by r^*. Next, the form of the metric h implies that the inverse matrix has coefficients $h^{ij} = \delta^{ij} + O(r^2)$. One thus obtains the desired estimates by taking the h- and δ-norms of these expressions and making the necessary comparisons. The expressions for the mean curvature follow by first taking the h- and δ-traces and then taking norms. The calculations for the Laplacian are similar. $\qquad\square$

3.3 The Legendrian Lawlor neck

A *Lagrangian Lawlor neck* is an embedded special Lagrangian submanifold of $\mathbb{C}^n$ with the topology of a cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ that is asymptotic to a pair of transverse special Lagrangian planes. It was discovered by Lawlor [16,17], and it was shown that any given pair of transverse special Lagrangian planes Π_1 and Π_2 can be realized as the asymptotic planes of a Lagrangian Lawlor neck if and only if the *characteristic angles* of Π_1 and Π_2 satisfy the *angle criterion* as described in the introduction. A truncated and re-scaled Lawlor neck can be lifted from $\mathbb{C}^n$ into $\mathbb{C}^n \times \mathbb{R}$ by the standard Legendrian lifting procedure to produce an embedded, cylindrical Legendrian submanifold asymptotic to a pair of Legendrian planes. This object can then be embedded into the sphere as a Legendrian submanifold using the Legendrian normal coordinates. The resulting object is the *Legendrian Lawlor neck*. The purpose of this section is to define this object precisely and to compute good estimates of its mean curvature and other geometric quantities.

The Lagrangian Lawlor neck The discussion begins with the definition of the Lagrangian Lawlor neck, which proceeds as follows. First, let $A := (a_1, \ldots, a_n)$ be a vector of positive real numbers and let $P : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be the function given by

$$P(A, \lambda) := \frac{\left(1 + a_1\lambda^2\right) \cdots \left(1 + a_n\lambda^2\right) - 1}{\lambda^2}. \tag{13}$$

Next, set

$$\theta_k(A, \lambda) := \int_{-\infty}^{\lambda} \frac{\mathrm{d}s}{(\frac{1}{a_k} + s^2)\sqrt{P(A, s)}}. \tag{14}$$

It is easy to see that the integrals (14) are well-defined and converge as $\lambda \to \infty$. Moreover, $P(A, \lambda) = O(\lambda^{2n-2})$ and so $\theta_k(A, \lambda) = \theta_k(A) + O(\lambda^{-n})$ for large λ, where $\theta_k(A) := \lim_{\lambda \to \infty} \theta_k(A, \lambda)$. The embedding giving the Lagrangian Lawlor neck in $\mathbb{C}^n$ is now defined as follows.

Definition 21 *For every* $A = (a_1, \ldots, a_n) \in \mathbb{R}^n$, *with* $a_k > 0$ *for all k, the* Lagrangian Lawlor embedding *with parameter A is the map* $\hat{F}_A : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{C}^n$ *given by*

$$\hat{F}_A(\lambda, \mu) := \left(z^1(\lambda, \mu), \ldots, z^n(\lambda, \mu)\right)$$

where

$$z^k(\lambda, \mu) = \mu^k \sqrt{\frac{1}{a_k} + \lambda^2} \, \exp\left(\mathrm{i}\theta_k(A, \lambda)\right)$$

and $\mu = (\mu^1, \ldots, \mu^n) \in \mathbb{R}^n$ *satisfies* $\sum(\mu^k)^2 = 1$ *so that* $\mu \in \mathbb{S}^{n-1}$. *The* Lagrangian Lawlor neck *with parameter A is the submanifold* $\hat{N}_A := \hat{F}_A(\mathbb{R} \times \mathbb{S}^{n-1})$. *The truncated and re-scaled Lagrangian Lawlor neck with parameters* ε, R, A *is the submanifold* $\hat{N}_{\varepsilon, R, A} := \varepsilon \hat{F}_A\left([-R, R] \times \mathbb{S}^{n-1}\right)$.

Observe that the two ends of $\hat{N}_A$ are asymptotic to the real plane $\mathbb{R}^n \times \{0\}$ (when $\lambda \to -\infty$) and to the plane $D_A(\mathbb{R}^n \times \{0\})$, where D_A is the diagonal $n \times n$ matrix having the entries $\mathrm{e}^{\mathrm{i}\theta_1(A)}, \ldots, \mathrm{e}^{\mathrm{i}\theta_n(A)}$ on the diagonal (when $\lambda \to \infty$). Moreover, it can be shown that $\sum_{k=1}^{n} \theta_k(A) = 0 \bmod 2\pi$ so that $D_A \in SU(n+1)$.

Lifting to a Legendrian submanifold In order to lift the Lagrangian submanifold $\hat{N}_A$ of $\mathbb{C}^n$ to a Legendrian submanifold of $\mathbb{C}^n \times \mathbb{R}$, it is necessary to find a function $S_A(\lambda, \mu)$ so that $F_A(\lambda, \mu) := \left(\hat{F}_A(\lambda, \mu), S_A(\lambda, \mu) \right)$ defines a Legendrian embedding. If $\alpha_0 = \frac{1}{2} dt + \sigma_0$ is the contact form, then the condition $F_A^* \alpha_0 = 0$ implies that the required function must satisfy $dS_A(\lambda, \mu) = -2 \hat{F}_A^* \sigma_0$. One then easily calculates that the required function must be

$$S_A(\lambda) := -n \int_0^\lambda \frac{ds}{\sqrt{P(A, s)}} \tag{15}$$

up to the addition of a constant factor. It is important to note that $S_A(\lambda) = S_A + O(\log(\lambda))$ for large λ in dimension $n = 2$, and $S_A(\lambda) = S_A + O(\lambda^{-n+2})$ for large λ in dimension $n \geq 3$, where S_A is a constant depending only on A and the dimension.

Definition 22 *For every $A = (a_1, \ldots, a_n) \in \mathbb{R}^n$ with $a_k > 0$ for all k, the* Legendrian Lawlor embedding *with parameter A is the map $F_A : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{C}^n \times \mathbb{R}$ given by*

$$F_A(\lambda, \mu) := \left(\hat{F}_A(\lambda, \mu), S_A(\lambda) \right).$$

The Legendrian Lawlor neck *with parameter A is the submanifold $N_A := F_A(\mathbb{R} \times \mathbb{S}^{n-1})$.*

Although the Lagrangian Lawlor embedding into $\mathbb{C}^n$ can be scaled in a straightforward way, scaling the Legendrian Lawlor embedding into $\mathbb{C}^n \times \mathbb{R}$ requires handling the $\mathbb{R}$-coordinate and the $\mathbb{C}^n$-coordinates differently. Indeed, to preserve the contact form it is necessary to scale the Legendrian Lawlor embedding as $F_{\varepsilon, A}(\lambda, \mu) := \left(\varepsilon \hat{F}_A(\lambda, \mu), \varepsilon^2 S_A(\lambda) \right)$. The *truncated and re-scaled Legendrian Lawlor neck* with parameters ε, R, A is the submanifold $N_{\varepsilon, R, A} := F_{\varepsilon, A}\left([-R, R] \times \mathbb{S}^{n-1}\right)$.

Properties of the Legendrian Lawlor neck The following propositions gather the relevant properties of $N_{\varepsilon, R, A}$ that will be used in the sequel. The first of these propositions gives a more detailed picture of the asymptotics of $N_{\varepsilon, R, A}$. The analogous results for $\hat{N}_{\varepsilon, R, A}$ have been proved in [3, Theorem 6], and their extension to the present case follows easily from Proposition 4. Let $\Pi_0 := \text{span}_{\mathbb{R}}\{e_1, \ldots, e_n\}$ and $\Pi_A := \text{span}_{\mathbb{R}}\{e^{i\theta_1(A)} e_1, \ldots, e^{i\theta_n(A)} e_n\}$.

Proposition 23 *If $\varepsilon > 0$ is sufficiently small, there exist radii $R \gg 1$ and $r \ll 1$ satisfying $R = O(r/\varepsilon)$ so that the following hold.*

1. $\hat{N}_{\varepsilon, R, A} \subseteq B_r(0)$ in $\mathbb{C}^n$ and $N_{\varepsilon, R, A} \subseteq B_r(0) \times (0, \varepsilon^2 S_A)$ in $\mathbb{C}^n \times \mathbb{R}$.
2. $N_{\varepsilon, R, A} \setminus B_{r/2}(0)$ *consists of two disconnected components projecting onto $\Pi_0 \cap Ann(r/2, r)$ and $\Pi_A \cap Ann(r/2, r)$, respectively, by nearest-point projection. Furthermore, there exists a function $G_\varepsilon : Ann(r/2, r) \to \mathbb{R}$ so that*

$$N_{\varepsilon, R, A} \cap Ann(r/2, r) = \left(\Lambda_{G_\varepsilon} \cap Ann(r/2, r) \right) \bigcup \left(D_{A, \varepsilon^2 S_A} \left(\Lambda_{G_\varepsilon} \right) \cap Ann(r/2, r) \right) \tag{16}$$

in the notation of Proposition 4, where $D_{A, \varepsilon^2 S_A}$ is the diagonal $SU(n)$-rotation bringing Π_0 to Π_A followed by upward translation by $\varepsilon^2 S_A$ in the Reeb direction.
3. *The function G_ε exhibits the scaling behaviour $G_\varepsilon(x) = \varepsilon^2 G_1(x/\varepsilon)$, and if $x \in Ann(r/2, r)$, then it satisfies the estimates*

$$\|\nabla^k G_\varepsilon(x)\| = O\left(\frac{\varepsilon^n}{\|x\|^{n-2+k}} \right) \tag{17}$$

for all $n \geq 2$ and $k \geq 0$, except $n = 2$ and $k = 0$ with respect to the metric induced by the ambient Euclidean metric. In the exceptional case, the estimate is $|G_\varepsilon(x)| = O(\varepsilon^2 |\log \|x\||)$.

The next proposition derives the intrinsic and extrinsic geometry of $N_{\varepsilon,R,A}$ induced by the metric g_0. Recall that this is the same as the geometry of $\hat{N}_{\varepsilon,R,A} \subseteq \mathbb{C}^n$ that is induced by the metric $\hat{g}_0$, and that this metric is close to being Euclidean sufficiently close to the origin. The formulæ below all result from straightforward computations.

Lemma 24 *The induced metric of the truncated Lagrangian Lawlor neck $\hat{N}_{\varepsilon,R,A}$ with respect to the Euclidean metric of $\mathbb{C}^n$ is given by*

$$h_{\varepsilon,A} = \varepsilon^2 \left[\sum_{k=1}^n \left(\frac{(\mu^k)^2}{\frac{1}{a_k} + \lambda^2} \right) \left(\lambda^2 + \frac{1}{P(A,\lambda)} \right) d\lambda^2 + \sum_{k=1}^n \left(\frac{1}{a_k} + \lambda^2 \right) \left(d\mu^k \right)^2 \right]. \quad (18)$$

The second fundamental form of $\hat{N}_{\varepsilon,R,A}$ with respect to the Euclidean metric of $\mathbb{C}^n$ is given by

$$B_{\varepsilon,A} = \varepsilon^2 \left[\frac{\mathrm{Sym}(d\lambda \otimes g_S)}{\sqrt{P(A,\lambda)}} + b \, d\lambda^3 \right] \quad (19)$$

where g_S is the standard metric of $\mathbb{S}^{n-1}$ while $\mathrm{Sym}(d\lambda \otimes g_S)$ is the symmetrization of $d\lambda \otimes g_S$ and

$$b = \sum_{k=1}^n \frac{(\mu^k)^2}{\sqrt{P(A,\lambda)} \left(\frac{1}{a_k} + \lambda^2 \right)^2} \left[\lambda^2 + \frac{1}{P(A,\lambda)} - \left(\frac{1}{a_k} + \lambda^2 \right) \left(1 + \frac{\lambda P'(A,\lambda)}{2P(A,\lambda)} \right) \right].$$

Finally, the mean curvature of $\hat{N}_{\varepsilon,R,A}$ with respect to the Euclidean metric of $\mathbb{C}^n$ vanishes.

A consequence of Lemma 24 and the methodology of Proposition 20 is the following estimate for the second fundamental form with respect to the ambient metric induced from the sphere.

Proposition 25 *The following estimates hold at a point (λ, μ) on $N_{\varepsilon,R,A,c}$. Here C is a constant independent of ε.*

- *The second fundamental form of $N_{\varepsilon,R,A,c}$ measured with respect to the metric g_0 satisfies*

$$\|B_{\varepsilon,A}\| + \varepsilon(1+\lambda^2)^{1/2}\|\nabla B_{\varepsilon,A}\| + \varepsilon^2(1+\lambda^2)\|\nabla^2 B_{\varepsilon,A}\|$$
$$\leq C \left[\frac{1}{\varepsilon(1+\lambda^2)^{(n+1)/2}} + \varepsilon(1+\lambda^2)^{1/2} \right].$$

- *The mean curvature of $N_{\varepsilon,R,A,c}$ measured with respect to the metric g_0 satisfies*

$$|H_{\varepsilon,A}| + \varepsilon(1+\lambda^2)^{1/2}\|\nabla H_{\varepsilon,A}\| + \varepsilon^2(1+\lambda^2)\|\nabla^2 H_{\varepsilon,A}\|$$
$$\leq C \left[\frac{\varepsilon}{(1+\lambda^2)^{(n-1)/2}} + \varepsilon(1+\lambda^2)^{1/2} \right].$$

Proof The computation of these estimates proceeds in two steps. First, one finds the estimate of the second fundamental form of $\hat{N}_{\varepsilon,R,A}$ with respect to Euclidean metric of $\mathbb{C}^n$. It suffices to take $A = (1, \ldots, 1)$ because it is always the case in this paper that the components of A lie

in a compact subset of the open positive quadrant of $\mathbb{R}^n$. Thus one can derive from Eqs. (18) and (19) that the coefficients of the metric and second fundamental form satisfy

$$h_{\varepsilon,A} = \varepsilon^2 \left[O(1) \, d\lambda^2 + (1 + \lambda^2) \, g_S \right]$$
$$B_{\varepsilon,A} = \varepsilon^2 \left[O\left((1 + \lambda^2)^{(1-n)/2} \right) \text{Sym}(d\lambda \otimes g_S) + O\left((1 + \lambda^2)^{(-1-n)/2} \right) d\lambda^3 \right].$$

Taking the norm of $B_{\varepsilon,A}$ with $h_{\varepsilon,A}$ as well as the norm of its first two covariant derivatives with respect to $h_{\varepsilon,A}$ yields the estimate

$$\|B_{\varepsilon,A}\| + \varepsilon(1 + \lambda^2)^{1/2} \|\nabla B_{\varepsilon,A}\| + \varepsilon^2(1 + \lambda^2)\|\nabla^2 B_{\varepsilon,A}\| \leq C\varepsilon^{-1}(1 + \lambda^2)^{-(n+1)/2}.$$

Then Proposition 20 and with the fact that the mean curvature with respect to the Euclidean metric of $\mathbb{C}^n$ is zero is used to derive the needed estimates with respect to the ambient metric coming from the sphere. Needed for this task is the fact that the ambient coordinate r is related to the neck coordinate λ by $r = \varepsilon(1 + \lambda^2)^{1/2}$.   □

3.4 Performing the connected sum

Let S_0, S_1 and S_2 be as in Sect. 3.1 and replace these by $\text{Pert}_\varepsilon(S_0)$, $\text{Pert}_{\varepsilon_1}(S_1)$ and $\text{Pert}_{\varepsilon_2}(S_2)$. It will now be shown how $\text{Pert}_\varepsilon(S_0)$ can be connected to $\text{Pert}_{\varepsilon_1}(S_1)$ and $\text{Pert}_{\varepsilon_2}(S_2)$ by choosing $\varepsilon, \varepsilon_1, \varepsilon_2$ well and inserting the appropriate Legendrian Lawlor necks.

Consider only the connected sum construction near the point $p_1 \in S_0$ and $e^{ic} p_1 \in S_1$, the construction at p_2 and $e^{-ic} p_2 \in S_2$ being similar. Choose Legendrian normal coordinates $\psi : \mathcal{U} \to \mathbb{C}^n \times \mathbb{R}$ in a neighbourhood $\mathcal{U}$ of p_1 and adapted to S_0 by choosing $\psi := \psi_{p_1, T_{p_1} S_0}$. Then $\psi(\text{Pert}_\varepsilon(S_0) \cap \mathcal{U})$ and $\psi(\text{Pert}_{\varepsilon_1}(S_1) \cap \mathcal{U})$ are graphical near the origin. More concretely, there is some fixed small number $r > 0$ along with functions $f_0 : B_r(0) \to \mathbb{R}$ and $f_1 : B_r(0) \to \mathbb{R}$ so that

$$\psi(\text{Pert}_\varepsilon(S_0) \cap \mathcal{U}) \cap B_r(0) = \Lambda_{f_0} \quad \text{and} \quad \psi(\text{Pert}_{\varepsilon_1}(S_1) \cap \mathcal{U}) \cap B_r(0) = D_{A,c}\left(\Lambda_{f_1}\right)$$

where Λ_f denotes the Legendrian submanifold generated by the function f and $D_{A,c}$ is rotation in the $\mathbb{C}^n$ coordinates by the diagonal matrix having $e^{i\theta_k(A)}$ on the diagonal, where $\theta_1(A), \ldots, \theta_n(A)$ are the asymptotic angles for some Legendrian Lawlor neck N_A, followed by translation by c in the $\mathbb{R}$ coordinate, as in the notation of Proposition 4. Moreover, by (11) and a change of variables, one can assume that the functions f_0 and f_1 can be expanded as

$$f_j(x) = \begin{cases} \dfrac{C_j \varepsilon_j^n}{\|x\|^{n-2}} + O(\varepsilon_j^n \|x\|^2) & n \geq 3 \\[2mm] \varepsilon_j^2 \left(C_j^0 + C_j^1 \log(\|x\|) \right) + O(\varepsilon_j^2 \|x\|^2) & n = 2 \end{cases} \tag{20}$$

in a small neighbourhood of the origin.

One can now attach Λ_{f_0} to Λ_{f_1} using a truncated and re-scaled Legendrian Lawlor neck as follows. Choose $\varepsilon, \varepsilon_1, \varepsilon_2$ to satisfy $\varepsilon^n C_0 = \varepsilon_1^n C_1 = \varepsilon_2^n C_2$ as well as the equation $c := \varepsilon^2 S_A$ (or $c := \varepsilon^2 S_A - \varepsilon_1^2 C_1^0$ in the $n = 2$ case). Next, set

$$r_\varepsilon := \varepsilon^s \quad \text{for} \quad s \in (0, 1)$$

to be the radius at which the gluing takes place. The parameter s will be chosen at the end of the proof; in the mean time, it suffices to know $\varepsilon \ll r_\varepsilon$ when ε is sufficiently small. Let $\eta : [0, 1] \to [0, 1]$ be a smooth, monotone cut-off function with support in $[0, 1/2]$ with

uniformly bounded derivatives. Define the following two functions

$$G_0(x) := \eta(\|x\|/r_\varepsilon)G_\varepsilon(x) + f_0(x)\,(1 - \eta(\|x\|/r_\varepsilon))$$
$$G_1(x) := \eta(\|x\|/r_\varepsilon)\,(G_\varepsilon(x) + \varepsilon^2 S_A) + (f_1(x) + c)\,(1 - \eta(\|x\|/r_\varepsilon))$$

where G_ε is the graphing function of the Legendrian Lawlor neck from Proposition 23 (with the appropriate modification for the $n = 2$ case). These two functions transition smoothly between the values of f_0 and $f_1 + \varepsilon^2 S_A$ outside $B_{r_\varepsilon}(0)$ and the values of G_ε inside $B_{r_\varepsilon/2}(0)$, respectively. Now define the submanifolds

$$T_0 := \Lambda_{G_0} \cap \mathrm{Ann}(r_\varepsilon/2, r_\varepsilon) \quad \text{and} \quad T_1 := D_A\left(\Lambda_{G_1}\right) \cap \mathrm{Ann}(r_\varepsilon/2, r_\varepsilon)$$

which transition smoothly between the Legendrian Lawlor neck $N_{\varepsilon,R,A,\varepsilon^2 S_A} \cap B_{r_\varepsilon/2}(0)$ and the submanifolds Λ_{f_0} and Λ_{f_1} outside of $B_{r_\varepsilon}(0)$, respectively. Finally, define the submanifold

$$\Lambda_\varepsilon := \left(\Lambda_{f_0} \cap \mathrm{Ann}(r_\varepsilon, r)\right) \cup T_0 \cup \left(N_{\varepsilon,R,A,\varepsilon^2 S_A} \cap B_{r_\varepsilon/2}(0)\right) \cup T_1$$
$$\cup \left(D_{A,\varepsilon^2 S_A}(\Lambda_{f_1}) \cap \mathrm{Ann}(r_\varepsilon, r)\right) \tag{21}$$

which combines these five pieces into a smooth, Legendrian submanifold of $\mathbb{C}^n \times \mathbb{R}$. Applying the inverse of the Legendrian normal coordinates then places this submanifold back in $\mathbb{S}^{2n+1}$.

Definition 26 *Choose $\varepsilon, \varepsilon_1, \varepsilon_2$ as above. The* Legendrian connected sum *of S_0 and S_1 is*

$$LCS_\varepsilon(S_0, S_1) := \left[\mathrm{Pert}_\varepsilon(S_0) \cup \mathrm{Pert}_{\varepsilon_1}(S_1) \backslash \psi^{-1}(B_r(0))\right] \cup \psi^{-1}(\Lambda_\varepsilon).$$

Call $\mathcal{T} := \psi^{-1}(T_0) \cup \psi^{-1}(T_1)$ the transition region *of $LCS_\varepsilon(S_0, S_1)$.*

The estimates of the geometry of $LCS_\varepsilon(S_0, S_1)$ are provided by Proposition 25 in the neck region and Proposition 17 in the exterior region. The following proposition gives the needed estimates in the transition region.

Proposition 27 *The following estimates hold in the transition region $\mathcal{T}$ of $\mathrm{LCS}_\varepsilon(S_0, S_1)$.*

- *The second fundamental form satisfies*

$$\|B\|_{C^2(\mathcal{T})}^* \leq C\left(\frac{\varepsilon^n}{r_\varepsilon^{n+1}} + r_\varepsilon\right)$$

  for some constant C independent of ε.

- *The mean curvature satisfies*

$$|\nabla \cdot H|_{C^1(\mathcal{T})}^* \leq C\left(\frac{\varepsilon^{n+1}}{r_\varepsilon^{n+3}} + 1\right)$$

$$\|H\|_{C^2(\mathcal{T})}^* \leq C\left(\frac{\varepsilon^n}{r_\varepsilon^{n-1}} + r_\varepsilon\right)$$

  for some constant C independent of ε.

- *The Laplacian satisfies*

$$\Delta(u) = \mathring{\Delta}(u) + O\left(\frac{\varepsilon^{2n}}{r_\varepsilon^{2n}} + r_\varepsilon^2\right) \cdot \mathring{\nabla}^2 u + O\left(\frac{\varepsilon^{2n}}{r_\varepsilon^{2n+1}} + r_\varepsilon\right) \cdot \mathring{\nabla} u$$

  where $O(r^*) \cdot$ denotes a linear operation on tensors with coefficients bounded by $O(r^*)$.

Proof The computation of these estimates will be carried out in Legendrian normal coordinates centered on the point of gluing. One begins by performing the calculations for the Lagrangian projection $[\mathrm{LCS}_\varepsilon(S_0, S_1)] \subseteq \mathbb{C}^n$ using the ambient Euclidean metric. The conclusion then follows from Theorem 8 and Proposition 20 as before.

Suppose now that $f := \eta f_1 + (1 - \eta) f_2$ where f_1 is the graphing function for the image of $[\mathrm{Pert}_\varepsilon(S_0)]$ in the Legendrian normal coordinate chart being used here, and f_2 is the graphing function for the end of the Lagrangian Lawlor neck that connects to it. Let η be the cut-off function appearing in the construction of $\mathrm{LCS}_\varepsilon(S_0, S_1)$. Set

$$L_{f_*} := \{(x, \overset{\circ}{\nabla} f_*(x)) : x \in B_{r_\varepsilon}(0) \backslash B_{r_\varepsilon/2}(0)\}$$

where $\overset{\circ}{\nabla}$ is the Euclidean gradient and $*$ refers to 1, 2 or nothing. Since the components of the second fundamental form of L_f with respect to the Euclidean metric are $B_{ijk} = f_{,ijk}$ then

$$\|B\| \leq \|\overset{\circ}{\nabla}^3 f_1\| + \|\overset{\circ}{\nabla}^3 f_2\| + \|T\|$$

where T is a tensor of order three whose components are linear combinations of the quantities $\eta_{,i}(f_{1,jk} - f_{2,jk})$, $\eta_{,ij}(f_{1,k} - f_{2,k})$ and $\eta_{,ijk}(f_1 - f_2)$ with $O(1)$ coefficients. To proceed, recall the expansions (17) and (20) which yield

$$\|\overset{\circ}{\nabla}^k f_j(x)\| = O\left(\frac{\varepsilon^n}{\|x\|^{n-2+k}}\right) \qquad \text{and} \qquad \|\overset{\circ}{\nabla}^k (f_1 - f_2)(x)\| = O\left(\frac{\varepsilon^{n+1}}{\|x\|^{n-1+k}}\right) + O(\varepsilon^n \|x\|^2).$$

Hence with $r_\varepsilon/2 \leq \|x\| \leq r_\varepsilon$, one finds

$$\|B\|_{C^0(\mathcal{T})} \leq C\left(\frac{\varepsilon^n}{r_\varepsilon^{n+1}} + \frac{\varepsilon^{n+1}}{r_\varepsilon^{n+2}} + \frac{\varepsilon^n}{r_\varepsilon}\right).$$

The estimates of the higher derivatives of B are similar, though more involved. Using Proposition 20 and the fact that $\varepsilon < r_\varepsilon$, the desired estimate follows.

Next, the divergence of the mean curvature of L_f with respect to the background Euclidean metric is $\nabla \cdot H = h^{kl} h^{ij} (f_{,ijk})_{;l}$ where h^{ij} are the coefficients of the inverse of the induced metric $h_{ij} := \delta_{ij} + \sum_s f_{,is} f_{,js}$ and the covariant derivatives are taken with respect to this metric. The Christoffel symbols are $\Gamma_{ijk} := \sum_s f_{,ijs} f_{,ks}$. Expanding in terms of f_1 and f_2 yields after some work

$$|\nabla \cdot H| \leq |\nabla_1 \cdot H_1| + |\nabla_2 \cdot H_2| + \|T'\| + O(\|\overset{\circ}{\nabla}^2 f\|^2 \|\overset{\circ}{\nabla}^4 f\|) + O(\|\overset{\circ}{\nabla}^3 f\|^2 \|\overset{\circ}{\nabla}^2 f\|)$$

where $\nabla_j \cdot H_j$ is the divergence of the mean curvature of L_{f_j} with respect to the induced metric of L_{f_j} with respect to the background Euclidean metric and this time T' is a tensor of order four whose components are linear combinations of the quantities $\eta_{,i}(f_{1,jkl} - f_{2,jkl})$, $\eta_{,ij}(f_{1,kl} - f_{2,kl})$, $\eta_{,ijk}(f_{1,l} - f_{2,l})$ and $\eta_{,ijkl}(f_1 - f_2)$ with $O(1)$ coefficients. Taking $r_\varepsilon/2 \leq \|x\| \leq r_\varepsilon$ now gives

$$|\nabla \cdot H|_{C^0(\mathcal{T})} \leq C\left(\frac{\varepsilon^{3n}}{r_\varepsilon^{3n+2}} + \frac{\varepsilon^{n+1}}{r_\varepsilon^{n+3}} + \frac{\varepsilon^n}{r_\varepsilon^2}\right)$$

using the estimate $|\nabla_1 \cdot H_1|_{*C^1(\mathcal{T})} \leq C\varepsilon^{3n} r_\varepsilon^{-3n-2}$ from Proposition 17 and the fact that $H_2 = 0$. The second derivatives of H can be estimated in the same way. Combined with the second fundamental form estimate above along with Proposition 20 and the fact that $\varepsilon < r_\varepsilon$, the first of the desired estimates follows. The estimate of $\|H\|_{C^2(\mathcal{T})}^*$ follows in a similar way to the previous two estimates, though yields a result that is not quite as good

because the leading term is the derivative of the Laplacian of the function G which is of size $O(\|\mathring{\nabla} G\|) = O(\varepsilon^n r_\varepsilon^{-n+1})$.

Finally, the Laplacian is $\Delta(u) := h^{ij}\left(u_{,ij} - \Gamma_{ij}^s u_{,s}\right)$ so that expanding as before yields

$$\Delta(u) = \mathring{\Delta}(u) + O(\|\mathring{\nabla}^2 f\|^2)\mathring{\nabla}^2 u + O(\|\mathring{\nabla}^3 f\| \cdot \|\mathring{\nabla}^2 f\|) \cdot \mathring{\nabla} u.$$

The desired estimate follows by estimating $\mathring{\nabla}^k f$ as before and invoking Proposition 20. □

4 The approximate solutions

4.1 The initial configurations

Examples of initial configurations of Legendrian n-spheres to which the Main Theorem applies will now be given. Denote the standard real basis of $\mathbb{C}^{n+1}$ by $P_1 := (1, \ldots, 0) \ldots$ $P_{n+1} := (0, \ldots, 1)$ in what follows, treating these as points in S_0 or as vectors in $\mathbb{C}^{n+1}$ depending on the context.

Example 1 The first example of an initial configuration can be described as follows. Choose complex numbers $\xi_s := \exp(2\pi i/m_s)$ for $s = 2, \ldots, n$ and $\bar{\xi}_{n+1} := \exp(2\pi i \sum_{s=2}^n 1/m_s)$, where m_s are odd integers with $|m_s| \geq 4$. Define U to be

$$U := \begin{pmatrix} 1 & & & \\ & \xi_2 & & \\ & & \ddots & \\ & & & \xi_{n+1} \end{pmatrix}$$

Next, let $N := \text{l.c.m.}(m_2, \ldots m_n)$, choose k and a odd satisfying g.c.d.$(a, kN) = 1$ and define $\zeta := \exp(2\pi i a/kN)$. Then $(\zeta U)^{kN} = I$ and no power of ζU smaller than kN is the identity.

The $U^s(S_0)$ all intersect at the points $\pm P_1$ so that the $(\zeta U)^s(S_0)$ acquire Hopf separation on the order of $\text{Arg}(\zeta)$. In fact, if $s_1 := (kN + 1)/2$ and $s_2 := (kN - 1)/2$ then $(\zeta U)^{s_1}(P_1) = \zeta^{1/2} P_1$ and $(\zeta U)^{s_2}(P_1) = \zeta^{-1/2} P_1$ with Hopf separation $\pm \pi a/kN$. Moreover $(\zeta U)^s(S_0)$ is closest to $(\zeta U)^{s_1}(S_0)$ and $(\zeta U)^{s_2}(S_0)$ at these points. Note that $T_{P_1} U^{s_1}(S_0) = \text{span}_{\mathbb{R}} \left\{ \xi_2^{1/2} P_2, \ldots, \xi_{n+1}^{1/2} P_{n+1} \right\}$ and $T_{-P_1} U^{s_2}(S_0) = \text{span}_{\mathbb{R}} \left\{ \xi_2^{-1/2} P_2, \ldots, \right.$ $\left. \xi_{n+1}^{-1/2} P_{n+1} \right\}$ and it is clear that these tangent planes are always transverse. One can also check that the angle criterion holds for these tangent spaces for sufficiently large choices of the m_s, depending on the dimension. The gluing should be performed by connecting the points given above. One can check that the resulting submanifold is closed and embedded (because the Hopf separation between any pair of n-spheres is never smaller $\pi a/kN$). The Legendrian angle function advances by $\pi a/kN$ from one n-sphere to the next and acquires a period of $2\pi a$ around the entire configuration.

Example 2 The next example is different from the first, in that gluing occurs at a pair of non-antipodal points. Once again, choose complex numbers $\xi_s := \exp(2\pi i/m_s)$ for $s = 2, \ldots, n$ but now $\bar{\xi}_{n+1} := -\exp\left(2\pi i \sum_{s=2}^n 1/m_s r\right)$, where m_s are integers with $|m_s| \geq 4$. Define U to be

$$U := \begin{pmatrix} 0 & \xi_2 & & & \\ 1 & 0 & & & \\ \hline & & \xi_3 & & \\ & & & \ddots & \\ & & & & \xi_{n+1} \end{pmatrix}$$

Next, let $N :=$ l.c.m.$(m_2, \ldots m_n)$, choose k and a satisfying g.c.d.$(a, kN) = 1$ and define $\zeta := \exp(2\pi i a/kN)$. Then $(\zeta U)^{2kN} = I$ and no power of ζU smaller than $2kN$ is the identity.

In the present case, $U(S_0) \cap S_0 = \{\pm P_2\}$ and $U^{-1}(S_0) \cap S_0 = \{\pm P_1\}$. Indeed, one finds that $U^{2s}(P_1) = \xi_2^s P_1 = U^{2s-1}(P_2)$ and $U^{2s} P_2 = \xi_2^s P_2 = U^{2s+1}(P_1)$. Multiplying by ζ creates the desired Hopf separation. In fact $\zeta^{2s} \xi_2^s P_1 \in (\zeta U)^{2s}(S_0)$ is closest to $\zeta^{2s-1} \xi_2^s P_1 \in (\zeta U)^{2s-1}(S_0)$ and $\zeta^{2s} \xi_2^s P_2 \in (\zeta U)^{2s}(S_0)$ is closest to $\zeta^{2s+1} \xi_2^s P_2 \in (\zeta U)^{2s+1}(S_0)$. The Hopf separation is $\pm 2\pi a/kN$, respectively. Note that $T_{P_1} S_0 = \mathrm{span}_{\mathbb{R}}\{P_2, \ldots, P_{n+1}\}$ and $T_{P_2} S_0 = \mathrm{span}_{\mathbb{R}}\{P_1, P_3, \ldots, P_{n+1}\}$ while $T_{P_2} U(S_0) = \mathrm{span}_{\mathbb{R}}\{\xi_2 P_1, \xi_3 P_3, \ldots, \xi_{n+1} P_{n+1}\}$ and $T_{P_1} U^{-1}(S_0) = \mathrm{span}_{\mathbb{R}}\{\bar{\xi}_2 P_2, \ldots, \bar{\xi}_{n+1} P_{n+1}\}$. It is clear that these tangent planes are always transverse. One can also can check that the angle criterion holds for these tangent spaces for sufficiently large choices of the m_s, depending on the dimension. The gluing should be performed by connecting the points given above. One can check that the resulting submanifold is closed but embedded only for appropriate choices of ξ_k and ζ. The Legendrian angle function acquires a period of $4\pi a$ around the entire configuration.

4.2 Assembling the approximate solutions

The approach outlined in the introduction for creating an approximately contact-stationary Legendrian submanifold from $U(n+1)$-rotated copies of S_0 will now be implemented here in the two examples just given. These examples will be constructed as follows.

1. Choose $U \in SU(n+1)$ as in Sect. 4.1.
2. Let $\zeta := e^{2\pi i a/kN}$ for some integer a and large integer k satisfying g.c.d.$(a, kN) = 1$.
3. Glue each $(\zeta U)^s(S_0)$ to its nearest neighbours at one of the points $(\zeta U)^s(\pm p)$ using the Legendrian connected sum procedure with scale parameter ε_k corresponding to the Hopf separation $2\pi a/kN$.

The *approximate solution* of the contact-stationary deformation equation is the submanifold constructed by means of the three steps above and will be denoted by $\tilde{\Lambda}_{U,\zeta}$.

The following terminology will be used in the sequel. Each $(\zeta U)^s(S_0)$ is attached to two neighbours at two distinct points chosen from the set $\{(\zeta U)^t(\pm p) : t = 0, \ldots, kN - 1\} \cap (\zeta U)^s(S_0)$. These points are the gluing points on $(\zeta U)^s(S_0)$ and these shall be denoted by p_{1s} and p_{2s}. (In the case $s = 0$, denote them simply by p_1 and p_2.) Now choose $r \in (0, r_0]$ where r_0 is sufficiently small but independent of ε, and define the disjoint union of balls centred on the gluing points

$$\mathcal{B}_r := \bigcup_{s=0}^{kN-1} \left[B_r\left((\zeta U)^s(p_{1s})\right) \cup B_r\left((\zeta U)^s(p_{2s})\right) \right] \cap \tilde{\Lambda}_{U,\zeta}. \tag{22}$$

Finally, subdivide $\tilde{\Lambda}_{U,\zeta}$ into regions of three distinct types: the union of all the neck regions, the union of all the transition regions and the union of all the exterior regions of $\tilde{\Lambda}_{U,\zeta}$ and denote these regions by $\mathcal{N}$, $\mathcal{T}$ and Λ'. That is, set

$$\mathcal{N} := \mathcal{B}_{r_\varepsilon/2} \quad \text{and} \quad \mathcal{T} := \mathcal{B}_{r_\varepsilon} \backslash \mathcal{B}_{r_\varepsilon/2} \quad \text{and} \quad \Lambda' := \tilde{\Lambda}_{U,\zeta} \backslash (\mathcal{N} \cup \mathcal{T}).$$

4.3 Symmetries satisfied by the approximate solutions

The approximate solution $\tilde{\Lambda}_{U,\zeta}$ constructed above is invariant with respect to the transformation ζU for two reasons. First, the collection of n-spheres $(\zeta U)^s(S_0)$ is invariant under

this group for obvious reasons. This shows that all of $\tilde{\Lambda}_{U,\zeta}$ except perhaps the neck and transition regions are invariant under this group. Second, since every neck region is uniquely determined by the tangent planes at the points of gluing and the Legendrian normal coordinates are equivariant with respect to ζU (in fact, ζU can be used to transplant the coordinates from the point p to every other gluing point), the neck regions are permuted amongst themselves by the action of ζU. Third, one can easily check that the Legendrian connected sum procedure keeps the transition regions of $\tilde{\Lambda}_{U,\zeta}$ invariant with respect to this group as well. There are also additional symmetries satisfied by $\tilde{\Lambda}_{U,\zeta}$ that will be important later on. These are specific to each of the two choices of U made above.

Example 1 Consider the transformations $K_j : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ defined by

$$K_j(z) := \left(z^1, z^2, \ldots, -z^j, \ldots, z^{n+1} \right) \quad \text{for } j = 1, \ldots, n+1. \tag{23}$$

These transformations are not $SU(n+1)$-rotations, but rather are orthogonal transformations of $\mathbb{C}^{n+1}$ (viewed as $\mathbb{R}^{2n+2}$) that still preserve the contact structure. One can easily verify that $K_j^{-1} \circ \zeta U \circ K_j = \zeta U$ for all j, so that each K_j descends to an isometry of the fundamental cell of $\tilde{\Lambda}_{U,\zeta}$ consisting of the spherical region, a transition region and a neck region.

Example 2 The transformations K_j for $3 \leq j \leq n+1$ defined in (23) from Example 1 are additional symmetries of $\tilde{\Lambda}_{U,\zeta}$ in this case as well. One additional symmetry involves the z^1 and z^2 coordinates. Define the transformations $K_{12} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ by

$$K_{12}(z^1, z^2, z^3, \ldots, z^{n+1}) := (-\bar{z}^2, -\bar{z}^1, \bar{z}^3, \ldots, \bar{z}^{n+1}).$$

Then it is easy to check that $K_{12}^{-1}(\zeta U)K_{12} = (\zeta U)^{-1}$ so that K_{12} descends to an isometry of the fundamental cell of $\tilde{\Lambda}_{U,\zeta}$.

5 Setting up the analysis

5.1 The Banach space inverse function theorem

The tasks ahead are to parametrize small contact deformations of $\tilde{\Lambda}_{U,\zeta}$ over a suitable Banach space of $C^{4,\beta}$ functions of $\tilde{\Lambda}_{U,\zeta}$ and solve the equation $\Phi_{\tilde{\Lambda}_{U,\zeta}}(f) = 0$, where $\Phi_{\tilde{\Lambda}_{U,\zeta}}$ is the contact-stationary deformation operator of $\tilde{\Lambda}_{U,\zeta}$ as defined abstractly in Definition 11. Henceforth, denote $\Phi_{U,\zeta} := \Phi_{\tilde{\Lambda}_{U,\zeta}}$ and denote also the linearization of this operator at zero by $\mathcal{L}_{U,\zeta}$.

The theorem that will be invoked to find f, albeit in a slightly subtle way, is the *Banach space inverse function theorem*. This fundamental result will now be stated in fairly general terms [1].

Theorem (IFT) *Let* $\Phi : X \to Z$ *be a smooth map of Banach spaces, set* $\Phi(0) := E$ *and denote the linearized operator* $D\Phi(0)$ *by* $\mathcal{L}$. *Suppose that* $\mathcal{L}$ *is bounded, surjective, and possesses a right inverse* $\mathcal{R} : Z \to X$ *satisfying the estimate*

$$\|\mathcal{R}(z)\| \leq C\|z\| \tag{24}$$

for all $z \in Z$. *Choose* R *so that if* $y \in B_R(0) \subseteq X$, *then*

$$\|\mathcal{L}(x) - D\Phi(y)(x)\| \leq \frac{1}{2C}\|x\| \tag{25}$$

for all $x \in X$. *Then if* $z \in Z$ *is such that*

$$\|z - E\| \le \frac{R}{2C},$$ (26)

there exists a unique $x \in B_R(0)$ *so that* $\Phi(x) = z$. *Moreover,* $\|x\| \le 2C\|z - E\|$.

5.2 Weighted Schauder norms

A preliminary step in applying the Banach space inverse function theorem is to have norms
for the Banach spaces X and Z which make explicit the dependence on the parameter k in
all the estimates (actually, on ε_k since this is more geometric). The reason this is necessary
is because when $k \to \infty$ and $\varepsilon_k \to 0$, then the approximate solution becomes singular
in each neck region and the contact-stationary Legendrian equation degenerates, making it
impossible to invoke the Banach space inverse function with more 'conventional' norms. In
the sequel, denote $\varepsilon := \varepsilon_k$.

The ε-dependence of the estimates will be tracked using *weighted Schauder norms* for the
spaces $C^{l,\beta}(\tilde{\Lambda}_{U,\zeta})$. To define the *weight function*, first define a 'regularized' distance function
ρ_0 on Λ' by requiring $\rho_0(x) := \operatorname{dist}(x, p_{js})$ in a neighbourhood of each of the gluing points
p_{js} and allowing ρ_0 to transition smoothly to one outside these neighbourhoods. Here the
terminology from Sect. 4.2 for the various regions of $\tilde{\Lambda}_{U,\zeta}$ is being used. Now make the
following definition.

Definition 28 *Define the* weight function $\rho_\varepsilon : \tilde{\Lambda}_{U,\zeta} \to \mathbb{R}$ *by*

$$\rho_\varepsilon(x) := \begin{cases} \rho_0(x) & x \in \Lambda' \\ Interpolation & x \in \mathcal{T} \\ \varepsilon\sqrt{1 + \lambda^2} & x = \psi_s^{-1}(\lambda, \mu) \in \mathcal{N} \end{cases}$$ (27)

where ψ_s *is the Legendrian normal coordinate chart for the neck region connecting* $(\zeta U)^s(S_0)$
to $(\zeta U)^{s+1}(S_0)$. *Furthermore, one can assume that the function* ρ_ε *is invariant under all
symmetries satisfied by* $\tilde{\Lambda}_{U,\zeta}$.

Now set $\Lambda'' := \{x \in \Lambda' : \rho_0(x) = 1\}$. Let q be any tensor on $\tilde{\Lambda}_{U,\zeta}$ and let $\mathcal{O} \subseteq \tilde{\Lambda}_{U,\zeta}$ be
any open subset. Recall the notation

$$\|q\|_{0,\mathcal{O}} = \sup_{x \in \mathcal{O}} \|q(x)\| \quad \text{and} \quad [q]_{\beta,\mathcal{O}} = \sup_{x,y \in \mathcal{O}} \frac{\|q(x) - PT(q(y))\|}{\operatorname{dist}(x, y)^\beta},$$

where the norms and the distance function that appear are taken with respect to the induced
metric of $\tilde{\Lambda}_{U,\zeta}$, while PT is the parallel transport operator from x to y with respect to this
metric. Now choose some large r_0 so that $x \in \Lambda''$ whenever x has a distance larger than r_0
from any of the gluing points. For any $0 < r \le 2r_0$, define the annular region $A_r := \mathcal{B}_r \setminus \mathcal{B}_{r/2}$
as well as the norm on any subset $\mathcal{O} \subseteq A_r$

$$|f|_{l,\beta,\gamma,\mathcal{O} \cap A_r} := \rho_\varepsilon^{-\gamma}(r)|f|_{0,\mathcal{O} \cap A_r} + \cdots + \rho_\varepsilon^{-\gamma+l}(r)\|\nabla^l f\|_{0,\mathcal{O} \cap A_r}$$
$$+ \rho_\varepsilon^{-\gamma+l+\beta}(r)[\nabla^l f]_{\beta,\mathcal{O} \cap A_r}.$$

Again, the derivatives which appear here are taken with respect to the induced metric of
$\tilde{\Lambda}_{U,\zeta}$. The weighted Schauder norm is now defined as follows.

Definition 29 *Let $\mathcal{O} \subset \tilde{\Lambda}_{U,\zeta}$. The $C_\gamma^{l,\beta}$ norm on $\mathcal{O} \subseteq \tilde{\Lambda}_{U,\zeta}$ is given by*

$$|f|_{C_\gamma^{l,\beta}(\mathcal{O})} := \sum_{i=0}^{l} \|\nabla^i f\|_{0,\mathcal{O}\cap\Lambda''} + [\nabla^l f]_{\beta,\mathcal{O}\cap\Lambda''} + \sup_{r\in(0,2r_0]} |f|_{l,\beta,\gamma,\mathcal{O}\cap\mathcal{A}_r}. \tag{28}$$

The notation for this norm will be abbreviated $|\cdot|_{C_\gamma^{l,\beta}}$ when there is no cause for confusion.

5.3 Legendrian deformations and the contact-stationary Legendrian equation

A second preliminary step is to have a concrete way of associating a Legendrian deformation ϕ_f of $\tilde{\Lambda}_{U,\zeta}$ to every function f in X. The key is to construct the association 'by hand' in such a way to have explicit control of ϕ_f in terms of f. Broadly speaking, the deformation associated to f is obtained as follows: one first extends f to a neighbourhood of $\tilde{\Lambda}_{U,\zeta}$ in a canonical way and multiplies by a suitable cut-off function; then one integrates the contact vector field generated by this extension up to time one.

Here are the details of this construction along with the necessary estimates. Observe first that the exponential map $\exp : N\tilde{\Lambda}_{U,\zeta} \to \mathbb{S}^{2n+1}$ of the normal bundle of $\tilde{\Lambda}_{U,\zeta}$ is a diffeomorphism in a tubular neighbourhood $\mathcal{V}$ of width $O(1)$ in $\mathcal{T} \cup \Lambda''$ and of width transitioning to $O(\varepsilon)$ in the narrowest part of $\mathcal{N}$. Let η be a smooth, monotone cut-off functions with support in $\mathcal{V}$ and define the extension operator $E_\varepsilon : C^{4,\beta}(\tilde{\Lambda}_{U,\zeta}) \to C^{4,\beta}(\mathbb{S}^{2n+1})$ by

$$E_\varepsilon(f)(x) := \begin{cases} f(\exp^{-1}(x)) \cdot \eta\left(\dfrac{\mathrm{dist}(x, \tilde{\Lambda}_{U,\zeta})}{\rho_\varepsilon(\exp^{-1}(x))}\right) & x \in \mathcal{V} \\ 0 & x \notin \mathcal{V} \end{cases}$$

where ρ_ε is the weight function defined in Definition 28. Consequently, $E_\varepsilon(f)$ coincides with f on $\tilde{\Lambda}_{U,\zeta}$. Now recall that the function $E_\varepsilon(f)$ defines a contact vector field X_f by means of the equations

$$\alpha(X_f) = E_\varepsilon(f) \quad \text{and} \quad X_f \lrcorner \, \mathrm{d}\alpha = \mathrm{d}E_\varepsilon(f)\big|_{\Xi},$$

and that integrating X_f yields a one-parameter family of contactomorphisms $\phi_f^t : \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}$ that is normal to $\tilde{\Lambda}_{U,\zeta}$. The time-one flow of this family is simply denoted ϕ_f.

Definition 30 *Functions on $\tilde{\Lambda}_{U,\zeta}$ are associated to contactomorphisms via $f \mapsto E_\varepsilon(f) \mapsto X_f \mapsto \phi_f$.*

5.4 The strategy of the proof

The deformation procedure developed above can be fed into the abstract definition of $\Phi_{U,\zeta}$ and leads to the concrete partial differential equation that must be solved to find the contact-stationary Legendrian submanifold near $\tilde{\Lambda}_{U,\zeta}$. That is, a function $f \in C_\gamma^{4,\beta}(\tilde{\Lambda}_{U,\zeta})$ must be found so that

$$\Phi_{U,\zeta}(f) := \nabla \cdot \left(H_{\phi_f(\tilde{\Lambda}_{U,\zeta})} \lrcorner \, \mathrm{d}\alpha\big|_{\phi_f(\tilde{\Lambda}_{U,\zeta})}\right) = 0$$

where ϕ_f is the contactomorphism defined above.

The strategy that will henceforth be used to apply the Banach space inverse function theorem to $\Phi_{U,\zeta}$ will take into account two important observations. The first observation is that $\Phi_{U,\zeta}$ is equivariant with respect to the symmetries of $\tilde{\Lambda}_{U,\zeta}$ induced by global isometries

of $\mathbb{S}^{2n+1}$ preserving the Legendrian condition. That is, if $\sigma \in O(2n+2)$ is such an isometry then $\Phi_{U,\zeta}(f \circ \sigma) = \Phi_{U,\zeta}(f) \circ \sigma$. The second observation is that $\Phi_{U,\zeta}$ does not have full rank because the first variation formula (2) applied to the generator of a contact isometry implies that every Legendrian submanifold Λ satisfies

$$\int\limits_{\phi_f(\Lambda)} \Phi_{\Lambda}(f) \cdot q_H \big|_{\phi_f(\Lambda)} = 0$$

for every function f and every Hermitian, harmonic, homogeneous polynomial q_H generating the $U(n+1)$-rotations of $\mathbb{S}^{2n+1}$. Consequently, the image of $\Phi_{U,\zeta}$ is constrained by $(n+1)^2$ conditions. The correct interpretation of these conditions is to say that the graph $\{(f, \Phi_{U,\zeta}(f)) : f \in C_\gamma^{4,\beta}(\tilde{\Lambda}_{U,\zeta})\}$ is contained in the Banach submanifold $\{(f, u) : \int_{\phi_f(\tilde{\Lambda}_{U,\zeta})} u \cdot q_H \big|_{\phi_f(\tilde{\Lambda}_{U,\zeta})} = 0 \ \forall \ q_H\}$ of $C_\gamma^{4,\beta}(\tilde{\Lambda}_{U,\zeta}) \times C_{\gamma-2}^{0,\beta}(\tilde{\Lambda}_{U,\zeta})$. Therefore it suffices to show that the equation $\pi \circ \Phi_{U,\zeta}(f) = 0$ has a solution, where π is the L^2-projection to the orthogonal complement of the subspace spanned by the functions $q_H \big|_{\tilde{\Lambda}_{U,\zeta}}$. Note that the linearization of $\pi \circ \Phi_{U,\zeta}$ at zero maps into this orthogonal complement, and thus $D\left(\pi \circ \Phi_{U,\zeta}\right)(0) = \mathcal{L}_{U,\zeta}$ holds.

These observations suggest that the Banach space inverse function theorem should be applied to the following spaces. In what follows, the subscript sym denotes invariance with respect to the group of symmetries of $\tilde{\Lambda}_{U,\zeta}$ from Sect. 4.3, so that if u belongs to a sym-subscripted space then $u \circ \sigma = u$ for all such symmetries σ. Also, the superscript $\perp$ denotes the L^2-orthogonal complement of a subspace of functions. Let $\mathcal{Q}_0 := \text{span}_{\mathbb{R}} \left\{ q_H \big|_{\tilde{\Lambda}_{U,\zeta}} \right\}$.

Definition 31 $X := \left[C_\gamma^{4,\beta}(\tilde{\Lambda}_{U,\zeta}) \right]_{\text{sym}}$ and $Z := \left[C_{\gamma-4}^{0,\beta}(\tilde{\Lambda}_{U,\zeta}) \cap \mathcal{Q}_0^{\perp} \right]_{\text{sym}}$.

By the two observations above, the operator $\pi \circ \Phi_{U,\zeta} : X \to Z$ is well-defined and it is sufficient to find a solution of the equation $\pi \circ \Phi_{U,\zeta}(f) = 0$ in these spaces.

6 The linear analysis

6.1 The Jacobi fields of the approximate solution

It is necessary to identify a subspace of functions that approximates the invariant Jacobi fields of $\mathcal{L}_{U,\zeta}$ in Z. As mentioned earlier, good approximating functions can be constructed as follows. One takes the exact Jacobi fields of the linearized operator of one of the spherical constituents of $\tilde{\Lambda}_{U,\zeta}$ and multiplies them by a cut-off function vanishing in a neighbourhood of the gluing points connecting this constituent to its neighbours. Such functions can be understood as generating transformations which act by $U(n+1)$-rotation of the constituent of $\tilde{\Lambda}_{U,\zeta}$ in question while leaving the others fixed. Then one chooses all ζU-invariant linear combinations of these functions.

More concretely, let $\chi_{\text{ext},r}$ be a smooth, monotone cut-off function that equals one in $\text{Pert}_\varepsilon(S_0) \setminus [B_r(p_1) \cup B_r(p_2)]$ for some r and transitions to zero elsewhere. Suppose χ_{ext} is invariant under the symmetries of $\tilde{\Lambda}_{U,\zeta}$. The functions just described are of the form

$$\tilde{q} := \left[\sum_{s=0}^{kN-1} \chi_{\text{ext}} \cdot q \big|_{\text{Pert}_\varepsilon(S_0)} \circ (\zeta U)^{-s} \right]$$

where q is any linear combination of the subset of polynomials in the list (10) that do not vanish on S_0. These functions, in coordinates $(x^1, x^2, \ldots, x^{n+1}) \in \mathbb{R}^{n+1}$ with $\sum_{s=1}^{n+1} (x^s)^2 = 1$, are

$$
\begin{aligned}
q_0(x) &:= 1 \\
q_1(x) &:= (n+1)(x^1)^2 - 1 \\
q_j(x) &:= (x^j)^2 - (x^{n+1})^2 \qquad \text{for } j = 2, \ldots, n \\
q_{\text{IM},jk}(x) &:= 2x^j x^k \qquad \text{for } 1 \le j < k \le n+1.
\end{aligned}
\tag{29}
$$

Note that there are $D := \frac{1}{2}(n+1)(n+2)$ such functions.

As mentioned earlier, one can only expect to find an appropriately bounded right inverse for $\mathcal{L}_{U,\zeta}$ on any subspace of functions that is 'sufficiently transverse' to the space of the invariant Jacobi fields. The following lemma establishes the necessary transversality condition for the space of approximate Jacobi fields defined above.

Lemma 32 *Suppose* $f : \tilde{\Lambda}_{U,\zeta} \to \mathbb{R}$ *is* L^2-*orthogonal to the functions* $q_1, \ldots q_D$ *and invariant with respect to all the symmetries of* $\tilde{\Lambda}_{U,\zeta}$. *Then* f *is 'approximately orthogonal' to the functions* $\tilde{q}_j$:

$$
\left| \int_{\tilde{\Lambda}_{U,\zeta}} f \cdot \tilde{q}_j \right| \le C r^{\gamma+n} |f|_{C_\delta^0}
$$

for all $j = 1, \ldots, D$.

Proof This estimate is derived in two steps. First, by invariance with respect to ζU, one finds

$$
\left| \int_{\tilde{\Lambda}_{U,\zeta}} f \cdot \tilde{q}_j \right| \le \left| \int_{\tilde{\Lambda}_{U,\zeta}} f \cdot \frac{1}{kN} \sum_{s=0}^{kN-1} q_j \circ (\zeta U)^s \right| + C r^{\gamma+n} |f|_{C_\delta^0}.
\tag{30}
$$

Now compute $\sum_{s=0}^{kN-1} q_j \circ (\zeta U)^s$ for each q_j. This calculation depends upon the specific choice of U made in Example 1 or 2. In the first example one finds:

$$
\begin{aligned}
\frac{1}{kN} \sum_{s=0}^{kN-1} q_0 \circ (\zeta U)^s(x) &= 1 \\
\frac{1}{kN} \sum_{s=0}^{kN-1} q_1 \circ (\zeta U)^s(x) &= (n+1)(x^1)^2 - 1 \\
\frac{1}{kN} \sum_{s=0}^{kN-1} q_j \circ (\zeta U)^s(x) &= \frac{(x^j)^2 - (x^{n+1})^2}{N} \qquad \text{for } j = 2, \ldots, n+1
\end{aligned}
\tag{31}
$$

whereas all others vanish. In the second example, it is better to use a slightly different basis for the space of functions spanned by $q_1, \ldots, q_D$. That is, take the matrices I, $H_3, \ldots, H_{n+1}$, H_{jk} and H'_{jk} from (10) and replace H_1 and H_2 by

$$
H'_1 = \begin{pmatrix} n-1 & & & \\ & n-1 & & \\ \hline & & -2 & \\ & & & \ddots \\ & & & & -2 \end{pmatrix} \quad \text{and} \quad H'_2 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ \hline & & 0 & \\ & & & \ddots \\ & & & & 0 \end{pmatrix}.
$$

The Hermitian, harmonic, homogeneous polynomials corresponding to these matrices, restricted to S_0, yield the functions $q_1'(x) = (n+1)\left((x^1)^2 + (x^2)^2\right) - 2$ and $q_2'(x) = (x^1)^2 - (x^2)^2$. One now finds

$$\frac{1}{kN}\sum_{s=0}^{kN-1} q_0 \circ (\zeta U)^s(x) = 1$$

$$\frac{1}{kN}\sum_{s=0}^{kN-1} q_1' \circ (\zeta U)^s(x) = (n+1)\left((x^1)^2 + (x^2)^2\right) - 2$$

$$\frac{1}{kN}\sum_{s=0}^{kN-1} q_j \circ (\zeta U)^s(x) = \frac{(x^j)^2 - (x^{n+1})^2}{N} \qquad \text{for } j = 3, \ldots, n+1 \tag{32}$$

$$\frac{1}{kN}\sum_{s=0}^{kN-1} q_{\text{IM},12} \circ (\zeta U)^s(x) = \frac{x^1 x^2}{N},$$

whereas all others vanish. Consequently, the integral term in (30) vanishes for q_j in the list (31) for Example 1 and for q_j in the list (32) for Example 2, and the estimate holds for these q_j.

In each example, the lists (31) and (32) do not form a basis for the space of functions spanned by $q_1, \ldots, q_D$. The remaining basis functions are

$$q_{\text{IM},jk} = x^j x^k \qquad \text{for } 1 \le j < k \le n+1$$

in Example 1, and

$$q_2 = (x^1)^2 - (x^2)^2$$
$$q_{\text{IM},1j} = x^2 x^j \qquad \text{for } 3 \le j \le n+1$$
$$q_{\text{IM},2j} = x^1 x^j \qquad \text{for } 3 \le j \le n+1$$

in Example 2. But for each of these functions, there is an additional symmetry $\sigma \in O(2n+2)$ under which $\tilde{\Lambda}_{U,\zeta}$ is invariant and $q \circ \sigma = -q$, as can be readily verified from the definition of the additional symmetries in Sect. 4.3. One can thus compute directly that

$$\int_{\tilde{\Lambda}_{U,\zeta}} f \cdot \tilde{q}_j = \int_{\tilde{\Lambda}_{U,\zeta}} (f \cdot \tilde{q}_j) \circ \sigma = - \int_{\tilde{\Lambda}_{U,\zeta}} f \cdot \tilde{q}_j = 0,$$

thereby verifying the estimate once again.   □

6.2 The linear estimate

This task at hand is to find an appropriately bounded right inverse for $\mathcal{L}_{U,\zeta} : X \to Z$ and a number of steps are needed to reach this goal. The necessary steps become more complicated as the dimension n decreases because of the nature of the indicial roots of the principal part of $\mathcal{L}_{U,\zeta}$, which is the bi-Laplacian operator, in low dimensions. The material below presents the simplest proofs available in three cases: for dimensions $n \ge 5$, for dimensions $n = 3, 4$ and for dimension 2. The theory underlying the results below is the theory of the Laplace operator on asymptotically flat manifolds and on punctured manifolds, which exists in several places in the literature. A particularly complete and focused source is [24].

The analysis begins with a lemma showing that the lower order term in the linearized operator $\mathcal{L}_{U,\zeta}$ (see Corollary 14) is uniformly small.

Lemma 33 *The operator* $Q_{U,\zeta} := 2\nabla \cdot B_{\tilde{\Lambda}_{U,\zeta}}(H_{\tilde{\Lambda}_{U,\zeta}}, \nabla u) - (H_{\tilde{\Lambda}_{U,\zeta}} \cdot \nabla)^2 u$ *satisfies the estimate*

$$|Q_{U,\zeta}(u)|_{C^{0,\beta}_{\gamma-4}} \leq C\left(\frac{\varepsilon^n}{r_\varepsilon^{n-2}} + \varepsilon^2 + r_\varepsilon^4\right)|u|_{C^{4,\beta}_\gamma}$$

for all $u \in X$.

Proof The operator $Q_{U,\zeta}$ satisfies the pointwise estimate

$$\left|[\rho_\varepsilon(x)]^{4-\gamma} Q_{U,\zeta}(u)(x)\right| \leq C[\rho_\varepsilon(x)]^2 \left|H_{\tilde{\Lambda}_{U,\zeta}}(x)\right| \left(\left\|B_{\tilde{\Lambda}_{U,\zeta}}(x)\right\| + \left\|\rho_\varepsilon(x)\nabla B_{\tilde{\Lambda}_{U,\zeta}}(x)\right\|\right)|u|_{C^{4,\beta}_\gamma}$$

for any $x \in \tilde{\Lambda}_{U,\zeta}$. If x belongs to the exterior region $\Lambda' \subseteq \tilde{\Lambda}_{U,\zeta}$, Proposition 17 gives the estimate

$$|Q_{U,\zeta}(u)|_{C^{0,\beta}_{\gamma-4}(\Lambda')} \leq C\varepsilon^{2n}r_\varepsilon^{-2n+2}|u|_{C^{4,\beta}_\gamma}.$$

If x belongs to the neck region $\mathcal{N} \subseteq \tilde{\Lambda}_{U,\zeta}$ where $\rho_\varepsilon(\lambda) = \varepsilon\sqrt{1+\lambda^2} := \rho(\lambda)$ Proposition 25 gives

$$\begin{aligned}|Q_{U,\zeta}(u)|_{C^{0,\beta}_{\gamma-4}(\mathcal{N})} &\leq \sup_{|\lambda|\leq r_\varepsilon/\varepsilon} C\,\varepsilon^2[\rho(\lambda)]^2\left(\frac{\varepsilon}{[\rho(\lambda)]^{n-1}} + \varepsilon\rho(\lambda)\right)\left(\frac{1}{\varepsilon[\rho(\lambda)]^{n+1}}\right.\\ &\quad \left. + \varepsilon\rho(\lambda)\right)|u|_{C^{4,\beta}_\gamma}\\ &\leq C\left(\varepsilon^2 + r_\varepsilon^4\right).\end{aligned}$$

Finally, if x belongs to the transition region $\mathcal{T} \subseteq \tilde{\Lambda}_{U,\zeta}$, Proposition 27 gives

$$\begin{aligned}|Q_{U,\zeta}(u)|_{C^{0,\beta}_{\gamma-4}(\mathcal{T})} &\leq C\,r_\varepsilon^2\left(\frac{\varepsilon^n}{r_\varepsilon^{n-1}} + r_\varepsilon\right)\left(\frac{\varepsilon^n}{r_\varepsilon^{n+1}} + r_\varepsilon\right)\\ &\leq C\left(\frac{\varepsilon^n}{r_\varepsilon^{n-2}} + r_\varepsilon^4\right)\end{aligned}$$

using the fact that $\varepsilon < r_\varepsilon$. Consolidating these separate results yields the desired estimate for the supremum norm. The Hölder norm estimate is similar. $\square$

Henceforth, set $L_{U,\zeta} := \Delta_{U,\zeta}(\Delta_{U,\zeta} + 2(n+1))$. The existence of an appropriately bounded right inverse in dimensions $n \geq 5$ is already a consequence of Lemma 33. It uses a fairly straightforward contradiction argument together with the fact that $\mathcal{L}_{U,\zeta}$ and $L_{U,\zeta}$ are self-adjoint.

Proposition 34 *Suppose* $n \geq 5$ *and choose* $\gamma \in (4-n, 0)$. *If* ε *is sufficiently small then there is a constant* C *independent of* ε *so that the operator* $\mathcal{L}_{U,\zeta} : X \cap \mathcal{Q}_0^\perp \to Z$ *is bijective and satisfies* $|u|_{C^{4,\beta}_\gamma} \leq C|\mathcal{L}_{U,\zeta}(u)|_{C^{0,\beta}_{\gamma-4}}$ *for all* $u \in X$ *where* C *is a constant independent of* ε.

Proof The strategy of this proof is as follows. It will be shown below that $L_{U,\zeta}$ is injective on $X \cap \mathcal{Q}_0^\perp$ and satisfies $|u|_{C^{4,\beta}_\gamma} \leq C_0|L_{U,\zeta}(u)|_{C^{0,\beta}_{\gamma-4}}$ for some constant C_0 independent of ε. Since $L_{U,\zeta}$ is self-adjoint then $L_{U,\zeta} : X \cap \mathcal{Q}_0^\perp \to Z$ is surjective. The estimate of Lemma 33

then shows that the operator $\mathcal{L}_{U,\zeta}$ is injective on $X \cap \mathcal{Q}_0^\perp$ as well when ε is sufficiently small, and satisfies the desired bound with $C := 2C_0$. Therefore $\mathcal{L}_{U,\zeta}$ is surjective, again by self-adjointness.

The injectivity bound for $L_{U,\zeta}$ can be found in this way. By standard elliptic theory and scaling arguments, there is already some constant C independent of ε so that

$$|u|_{C_\gamma^{4,\beta}} \leq C \left(|L_{U,\zeta}(u)|_{C_{\gamma-4}^{0,\beta}} + |u|_{C_\gamma^{0,\beta}} \right) \tag{33}$$

for all $u \in C_\gamma^{4,\beta}(\tilde{\Lambda}_{U,\zeta})$. Hence it is enough to obtain a contradiction from the assumption that the estimate $|u|_{C_\gamma^{0,\beta}} \leq C|L_{U,\zeta}(u)|_{C_{\gamma-4}^{0,\beta}}$ is false. In other words, suppose that there are sequences of:

- scale parameters $\varepsilon_i \to 0$ and corresponding complex numbers $\zeta_i \to 1$
- approximate solutions $\Lambda_i := \tilde{\Lambda}_{U,\zeta_i}$
- linear operators $L_i := L_{U,\zeta_i}$
- weight functions $\rho_i := \rho_{\varepsilon_i}$
- Banach spaces $X_i := \left[C_\gamma^{4,\beta}(\Lambda_i) \cap \mathcal{Q}_0^\perp \right]_{sym}$ and $Z_i := \left[C_{\gamma-4}^{0,\beta}(\Lambda_i) \cap \mathcal{Q}_0^\perp \right]_{sym}$

- functions $u_i \in X_i$ satisfying $|u_i|_{C_\gamma^{0,\beta}} = 1$ and $\lim_{i\to\infty} |L_i(u_i)|_{C_{\gamma-4}^{0,\beta}} = 0$.

Now let $p_i \in \Lambda_i$ be a point where $u_i(p_i)[\rho_i(p_i)]^{-\gamma} = 1$. There are two cases to consider.

Case 1: The non-concentrating case Let $S_R := S_0 \backslash [B_R(p_1) \cup B_R(p_2)]$ and $S_0^* := S_0 \backslash \{p_1, p_2\}$ where p_1, p_2 are the gluing points of S_0. Suppose without loss of generality that p_i converges to some point $p \in S_R$ for some non-zero radius R. By the estimate (33) and the Arzela–Ascoli theorem, there is a subsequence of u_i converging to a non-zero function $u_R \in C^{4,\beta}(S_R)$. It is now possible to pass to a further subsequence and obtain convergence to a non-zero function $u^* \in C_\gamma^{4,\beta}(S_0^*)$. This function satisfies $\Delta_S(\Delta_S + 2(n+1))(u^*) = 0$ on S_0^*, where Δ_S is the Laplacian of the standard sphere. Since $\gamma - 2 \in (2-n, 0)$, then $(\Delta_S + 2(n+1))(u^*)$ is actually smooth and thus must be constant. Since $\gamma \in (4-n, 0) \subseteq (2-n, 0)$, then u^* is also smooth and consequently must be a linear combination of the functions in the list (29). But the orthogonality and symmetry conditions that u^* must satisfy as a limit of functions $u_i \in X_i$ rules this out, as can be seen by allowing $\varepsilon_i \to 0$ in Lemma 32.

Case 2: The case where concentration occurs Suppose without loss of generality that p_i converges to the gluing point $p_1 \in S_0$. Consequently p_i eventually enters a Legendrian normal neighbourhood containing the neck region connecting S_0 to its neighbour at p_1. One can thus use the Legendrian normal coordinates and the Legendrian Lawlor embedding to write p_i as a point $(\lambda_i, \mu_i) \in \mathbb{R} \times \mathbb{S}^{n-1}$ and to consider $u_i \in C_\gamma^{4,\beta}(\mathbb{R} \times \mathbb{S}^{n-1})$. The norm on this space is a weighted norm, where the weight function is $\rho(\lambda) := \varepsilon\sqrt{1 + \lambda^2}$. Up to subsequences, there are now two possibilities: either $(\lambda_i, \mu_i) \to (\lambda, \mu)$ or else $\lambda_i \to \infty$. Each of these possibilities will be ruled out in turn. In the first of these possibilities, the estimate (33) and the Arzela–Ascoli theorem on the neck itself imply that u_i has a subsequence converging to a non-zero function $u \in C_{\gamma'}^{4,\beta}(\mathbb{R} \times \mathbb{S}^{n-1})$ for any $\gamma' > \gamma$ and satisfying the equation $\Delta_N(\Delta_N(u)) = 0$ where Δ_N is the Laplacian of the Lawlor neck. Since $\gamma \in (4-n, 0)$, this solution is decaying at infinity. But given the form of the Lawlor neck metric (18) on $\mathbb{R} \times \mathbb{S}^{n-1}$, there can be no decaying solutions of this equation. In the second possibility, one

can consider u_i on a ball of constant radius about λ_i and re-scale by a factor of ε_i^{-1} to obtain a sequence of functions which possesses a subsequence converging to a non-zero, decaying solution of $\mathring{\Delta}(\mathring{\Delta}(u)) = 0$ on $\mathbb{R}^n$ with respect to the Euclidean metric. Again, there are no decaying solutions of this equation.   □

Because of the overlapping nature of the indicial roots of the bi-Laplacian, the technique above is not available in the lower dimensions $n = 4, 3, 2$. In these cases, right inverses will have to be constructed 'by hand' and the required estimates derived as part of the construction. Two preliminary lemmata are needed.

Lemma 35 *Suppose $L_{U,\zeta} : X \to Z$ possesses a right inverse $\mathcal{R} : Z \to X$ satisfying the bound $|\mathcal{R}(f)|_{C_\gamma^{4,\beta}} \leq C|f|_{C_{\gamma-4}^{0,\beta}}$ where $C := C(\varepsilon)$ is a constant depending in some way on ε. If*

$$\limsup_{\varepsilon \to 0} C(\varepsilon)\|Q_{U,\zeta}\|_{\mathrm{op}} < \frac{1}{2}$$

and ε is sufficiently small, then $\mathcal{L}_{U,\zeta}$ also possesses a right inverse satisfying the same sort of bound, with $2C$ in place of C. Here $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm of a map between Banach spaces.

Proof Propose $u := \mathcal{R}(f + g)$ as the solution of the equation $\mathcal{L}_{U,\zeta}(u) = f$ so that g satisfies $g = -Q_{U,\zeta} \circ \mathcal{R}(f + g)$. The mapping $T_f : Z \to Z$ given by $T_f(g) := -Q_{U,\zeta} \circ \mathcal{R}(f + g)$ then satisfies

$$|T_f(g_1) - T_f(g_2)|_{C_{\gamma-4}^{0,\beta}} \leq C\|Q_{U,\zeta}\|_{\mathrm{op}}|g_1 - g_2|_{C_{\gamma-4}^{0,\beta}} \quad \text{and}$$

$$|T_f(g)|_{C_{\gamma-4}^{0,\beta}} \leq C\|Q_{U,\zeta}\|_{\mathrm{op}}|f + g|_{C_{\gamma-4}^{0,\beta}}$$

so that T_f is a contraction mapping on $B_R(0)$ when $|f|_{C_{\gamma-4}^{0,\beta}} < R$ and ε is sufficiently small. Therefore there is a unique fixed point g_f. The desired right inverse is the map $f \mapsto \mathcal{R}(f + g_f)$.   □

Lemma 36 *The following facts about the kernel of the Laplacian of the Lawlor neck are true.*

- *The harmonic functions of sub-linear growth are spanned by the constant function 1 and the function $S_A(\lambda)$ given in equation (15).*
- *Suppose $f = a_i^+ \mu^i |\lambda| + O(|\lambda|^{-1})$ for $\lambda \gg 1$ and $f = a_i^- \mu^i |\lambda| + O(|\lambda|^{-1})$ for $\lambda \ll -1$ in the coordinates of the neck, where $a_i^\pm \in \mathbb{R}$. Then there is a linearly growing harmonic function which equals f up to $O(|\lambda|^{-1})$ on the ends of the neck.*

Proof By separation of variables, one expects to find 2 linearly independent harmonic functions depending only on λ, as well as $2n$ linearly independent and linearly growing harmonic functions. The constant function 1 is clearly an example of the first kind. The Lagrangian analogue in $\mathbb{C}^n$ of the procedure of Corollary 15 can be used to find the remaining functions. (That is, a Hamiltonian function that generates a symplectic isometry and then restricted to a minimal Lagrangian submanifold gives a harmonic function on the submanifold because $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} H(\phi_{J\nabla f}(L)) = \mathcal{L}(f\big|_L)$ exactly as in the corollary.) The function S_A arises in this way by considering the infinitesimal generator of dilations. The linearly growing functions arise in this way by considering the infinitesimal generators of translations parallel to one or the other of the asymptotic n-planes of the neck. Since these n-planes are transverse, the $2n$ resulting functions are linearly independent and form a basis for linear growing functions on each end of the neck separately.   □

The estimates for the lower dimensions $n = 4, 3, 2$ can now be derived. Before beginning the proofs of the estimates, define the following smooth, monotone cut-off functions:

$$\chi_{\text{ext},r} := \begin{cases} 1 & x \in \mathcal{B}_r^c \\ \text{Interpolation} & x \in \mathcal{B}_r \backslash \mathcal{B}_{r/2} \\ 0 & x \in \mathcal{B}_{r/2} \end{cases}$$

$$\chi_{\text{neck},r} := 1 - \chi_{\text{ext},r}$$

where $\mathcal{B}_r$ is the disjoint union of balls of radius r containing the neck regions as defined in (22). One can assume that these functions are invariant with respect to the symmetries of $\tilde{\Lambda}_{U,\zeta}$.

Proposition 37 *Suppose* $n = 4$ *and choose* $\gamma \in (0, 1)$. *Then the operator* $L_{U,\zeta} : X \to Z$ *possesses a right inverse* $\mathcal{R} : Z \to X$ *satisfying the bound* $|\mathcal{R}(f)|_{C_\gamma^{4,\beta}} \leq C|f|_{C_{\gamma-4}^{0,\beta}}$ *where C is a constant independent of* ε.

Proof Using a contradiction argument as in the proof of Proposition 34, one can show that $\Delta_{U,\zeta} : [C_{\gamma-2}^{2,\beta}(\tilde{\Lambda}_{U,\zeta}) \cap \mathcal{Q}_0]_{\text{sym}} \to Z$ is bijective and satisfies the estimate $|w|_{C_{\gamma-2}^{2,\beta}} \leq C|\Delta_{U,\zeta}(w)|_{C_{\gamma-4}^{0,\beta}}$ for all $w \in [C_{\gamma-2}^{2,\beta}(\tilde{\Lambda}_{U,\zeta}) \cap \mathcal{Q}_0]_{\text{sym}}$ where C is a constant independent of ε. The reason the same argument works is because the choice $\gamma - 2 \in (-2, -1)$ implies that the Laplacian on $S_0 \backslash \{p_1, p_2\}$ and on N is bijective. Moreover, the self-adjointness of $\Delta_{U,\zeta}$ implies that the orthogonality conditions satisfied by f are inherited by $w := \Delta_{U,\zeta}^{-1}(f)$. Therefore it remains only to solve the equation $(\Delta_{U,\zeta} + 2(n+1))(u) = w$ for $u \in C_\gamma^{4,\beta}(\tilde{\Lambda}_{U,\zeta})$ and $w \in [C_{\gamma-2}^{2,\beta}(\tilde{\Lambda}_{U,\zeta}) \cap \mathcal{Q}_0]_{\text{sym}}$ for $\gamma \in (0, 1)$.

The technique that will be used to find a solution of $\left(\Delta_{U,\zeta} + 2(n+1)\right)(u_{\text{ext}}) = 0$ will be to patch together local solutions on the exterior regions with local solutions on the neck regions of $\tilde{\Lambda}_{U,\zeta}$ to construct a good approximate solution. This process will then be iterated to yield an exact solution. To begin the patching process, choose four radii satisfying $0 < r_3 \ll r_4 < r_2 < r_1$ in such a way that the supports of $\nabla \chi_{\text{ext},r_j}$ do not overlap. It will also be necessary to choose $r_3 = O(\varepsilon)$ and r_1, r_2, r_4 small a priori but independent of ε.

Step 1: The exterior regions Define the functions $\hat{q}_i := \sum_{s=0}^{kN-1} q_i\big|_{S_0} \circ (\zeta U)^s$ as well as $\hat{q}_i^{\text{approx}} := \hat{q}_i \chi_{\text{ext},r_1}$ and

$$w_{\text{ext}} := w \chi_{\text{ext},r_1} - \sum_{i,j=1}^{D} M^{ij} \left(\int_{S_0} w \chi_{\text{ext},r_1} \hat{q}_i \right) \hat{q}_j^{\text{approx}}$$

where M^{ij} is the inverse of the matrix whose coefficients are $\int_{S_0} \hat{q}_i \hat{q}_j^{\text{approx}}$. This is the L^2-projection of w_{ext} to the orthogonal complement of the space of functions spanned by $\hat{q}_1, \ldots, \hat{q}_D$. Furthermore, $w_{\text{ext}}\big|_{S_0}$ is a smooth function of compact support on S_0^* which is, by the calculations of Lemma 32, orthogonal to $q_1, \ldots, q_D$. The estimate of Lemma 32 implies $|w_{\text{ext}}|_{C_{\gamma-2}^{2,\beta}} \leq C|w|_{C_{\gamma-2}^{2,\beta}}$ with C independent of ε because $\gamma \in (0, 1)$. By ζU-invariance, it is necessary only to find the solution $u_{\text{ext}}\big|_{S_0}$ and then extend it by ζU-invariance to all of $\tilde{\Lambda}_{U,\zeta}$. By the theory of the Laplace operator on S_0 in smooth spaces, there exists a solution

$u_{\text{ext}} \in C^{4,\beta}(S_0)$ of the equation $(\Delta_S + 2(n + 1))(u) = w_{\text{ext}}$. By considering the Taylor expansion of u_{ext} at the points p_1 and p_2, one can write

$$u_{\text{ext}} := v_{\text{ext}} + a \left(\eta^1_{\text{neck},r_2} + \eta^2_{\text{neck},r_2} \right) \tag{34}$$

where $a := u_{\text{ext}}(p_1) = u_{\text{ext}}(p_2)$ (which are the same by symmetry) and $v_{\text{ext}} \in C^{4,\beta}_\gamma(S_0 \setminus \{p_1, p_2\})$ (since $\gamma \in (0, 1)$ and v_{ext} grows linearly with distance from each p_i). Here η^i_{neck,r_2} is a smooth, monotone cut-off function that equals one near p_i and transitions to zero elsewhere, and equals χ_{neck,r_2} on the region of overlap. Moreover, the estimate

$$|v_{\text{ext}}|_{C^{4,\beta}_\gamma(S_0 \setminus \{p_1, p_2\})} + |a| \le C |w_{\text{ext}}|_{C^{2,\beta}(S_0 \setminus \{p_1, p_2\})} \le C r_1^{\gamma - 2} |w_{\text{ext}}|_{C^{2,\beta}_{\gamma - 2}}$$

is valid for some constant C independent of ε.

At present, the function u_{ext} can be viewed as a function defined on $\tilde{\Lambda}_{U,\zeta} \setminus \mathcal{B}_r$ for some small r. To extend u_{ext} to a function defined on all of $\tilde{\Lambda}_{U,\zeta}$, first subtract aJ where J is the function in the list (31) in Example 1 and or in the list (32) in Example 2 which is even with respect to the symmetry exchanging p_1 and p_2 and has $J(p_i) = 1$. Note that on can write $J = (\eta^1_{\text{neck},r_2} + \eta^2_{\text{neck},r_2}) + \tilde{J}$ where $\tilde{J} \in C^{4,\beta}_\gamma(S_0)$. Then set

$$\bar{u}_{\text{ext}} := \chi_{\text{ext},r_2} \left(v_{\text{ext}} - a\tilde{J} \right).$$

Finally, one deduces the estimate

$$|\bar{u}_{\text{ext}}|_{C^{4,\beta}_\gamma} \le C|w|_{C^{2,\beta}_{\gamma - 2}}$$

where the constant C is independent of ε.

Step 2: The neck regions   Define the function

$$w_{\text{neck}} := \chi_{\text{neck},r_3} \left(w - (\Delta_{U,\zeta} + 2(n + 1))(\bar{u}_{\text{ext}}) \right)$$

and consider w_{neck} as a function of compact support defined on the ε-scaled Lawlor neck N_ε. One now looks for a solution of the equation $\Delta_N(u) = w_{\text{neck}}$ where Δ_N is the Laplacian of N_ε. Set $\gamma' := -2 - \gamma$. Since w_{neck} has compact support, then $w_{\text{neck}} \in C^{2,\beta}_{\gamma'-2}(N_\varepsilon)$ where the norm in this case is the weighted Hölder norm on N_ε with the weight function $\rho_\varepsilon(\lambda) := \varepsilon\sqrt{1 + \lambda^2}$ given in the coordinates of the Lawlor neck. By the theory of the Laplace operator on asymptotically flat manifolds, the operator

$$\Delta_N : C^{4,\beta}_{\gamma'}(N_\varepsilon) \oplus \mathcal{D} \to C^{2,\beta}_{\gamma'-2}(N_\varepsilon)$$

is surjective with a two-dimensional kernel. Here $\mathcal{D}$ is the *deficiency space* given by

$$\mathcal{D} := \text{span}_{\mathbb{R}} \left\{ \frac{\chi_{\text{ext},r_3}}{|\lambda|^2}, \chi_{\text{ext},r_3} \right\}.$$

The reason for this is because the range (γ', γ) contains rates of growth of the indicial roots of the Laplacian corresponding to constant functions and functions which decay like the Green's function at infinity, and the symmetry conditions require f to be even with respect to the transformation $\lambda \mapsto -\lambda$. Thus one can find a solution in this space, which has the form

$$u_{\text{neck}} := v_{\text{neck}} + b \frac{\chi_{\text{ext},r_3}}{|\lambda|^2}, \tag{35}$$

adding a constant if necessary, where $v_{\text{neck}} \in C^{4,\beta}_{\gamma'}(N_\varepsilon)$ and satisfies the estimates

$$|v_{\text{neck}}|_{C^{4,\beta}_{\gamma'}(N_\varepsilon)} + \varepsilon^{2+\gamma}|b| \leq Cr_3^{2+2\gamma}|w_{\text{neck}}|_{C^{2,\beta}_{\gamma-2}} \tag{36}$$

for some constant C independent of ε. This is because w_{neck} is supported in the region of radius $O(r_3)$. Finally, extend u_{neck} to all of $\tilde{\Lambda}_{U,\zeta}$ simply by setting

$$\bar{u}_{\text{neck}} := \chi_{\text{neck},r_4} u_{\text{neck}}$$

and symmetrizing with respect to the group generated by ζU. Finally, one deduces the estimate

$$|\bar{u}_{\text{neck}}|_{C^{4,\beta}_\gamma} \leq |v_{\text{neck}}|_{C^{4,\beta}_\gamma(N_\varepsilon)} + \sup_{r_3/\varepsilon \leq |\lambda| \leq r_4/\varepsilon} [\rho_\varepsilon(\lambda)]^{-\gamma}|b||\lambda|^{-2}$$

$$\leq C\left[\left(\frac{r_3}{\varepsilon}\right)^{2+2\gamma} + \left(\frac{r_3}{\varepsilon}\right)^\gamma\right]|w_{\text{neck}}|_{C^{2,\beta}_{\gamma-2}}$$

$$\leq C|w|_{C^{2,\beta}_{\gamma-2}}$$

using Eq. (36) and the estimate for $|\bar{u}_{\text{ext}}|_{C^{4,\beta}_\gamma}$ given in Step 1 along with the values of the radii r_i and the estimate for $|w_{\text{ext}}|_{C^{2,\beta}_{\gamma-2}}$ in terms of $|w|_{C^{2,\beta}_{\gamma-2}}$.

Step 3: Estimates and convergence Set $\bar{u} := \bar{u}_{\text{ext}} + \bar{u}_{\text{neck}}$. The estimates from Steps 1 and 2 show that $|\bar{u}|_{C^{4,\beta}_\gamma} \leq C|w|_{C^{2,\beta}_{\gamma-2}}$. The function $\bar{u}$ should be seen as the approximate solution of the equation $(\Delta_{U,\zeta} + 2(n+1))(u) = w$ satisfying the appropriate estimate. To justify this, it must be shown that $|(\Delta_{U,\zeta} + 2(n+1))(\bar{u}) - w|_{C^{2,\beta}_{\gamma-2}} \leq \frac{1}{2}|w|_{C^{2,\beta}_{\gamma-2}}$ when ε is sufficiently small. If this holds, then the procedure of Steps 1 and 2 can be iterated to yield an exact solution in the limit and satisfying the appropriate estimate.

A straightforward calculation shows that

$$(\Delta_{U,\zeta} + 2(n+1))(\bar{u}) - w = 2(n+1)\bar{u}_{\text{neck}}$$
$$+ (\Delta_{U,\zeta} - \Delta_N)(\bar{u}_{\text{neck}}) + \chi_{\text{ext},r_3}(\Delta_{U,\zeta} - \Delta_S)(\bar{u}_{\text{ext}})$$
$$+ [\Delta_N, \chi_{\text{neck},r_4}](u_{\text{neck}}) + \chi_{\text{ext},r_2}(w - w_{\text{ext}}). \tag{37}$$

where $[\Delta, \chi](u)$ is notation for $\Delta(\chi u) - \chi\Delta(u)$. The various terms in (37) are all small multiples of $|w|_{C^{2,\beta}_{\gamma-2}}$ when measured with respect to the $C^{2,\beta}_{\gamma-2}$ norm, but for different reasons.

- The term on the first line is supported in the region $r < r_4$. Thus its $C^{2,\beta}_{\gamma-2}$ norm acquires the factor r_4^2 so that $2(n+1)|\bar{u}_{\text{neck}}|_{C^{2,\beta}_{\gamma-2}} \leq Cr_4^2|w|_{C^{2,\beta}_{\gamma-2}}$. This is can be made small with a small enough initial choice of r_4.
- In the terms on the second line, the operators $\Delta_{U,\zeta} - \Delta_S$ and $\Delta_{U,\zeta} - \Delta_N$ are uniformly small where their arguments are supported, as can be deduced from Propositions 17, 20 and 25.
- In the term on the third line, the function u_{neck} has been engineered to have strong decay in the support of $\nabla\chi_{\text{neck},r_4}$. Indeed, the estimate (36) implies that $|v_{\text{neck}}|_{C^{2,\beta}_{\gamma-2}(\mathcal{B}_{r_4}\setminus\mathcal{B}_{r_4/2})} \leq C(r_3/r_4)^{2+2\gamma}$, Since $r_3/r_4 = O(\varepsilon)$, this quantity can be made as small as desired by choosing ε and the r_i small enough. A similar analysis holds for the $b\chi_{\text{ext},r_3}|\lambda|^{-2}$ term in $\bar{u}_{\text{neck}}$.
- The remaining term is handled by Lemma 32 and becomes small when r_1 is small enough.

These arguments show that $|(\Delta_{U,\zeta} + 2(n+1))(\bar{u}) - w|_{C^{2,\beta}_{\gamma-2}} \leq \frac{1}{2}|w|_{C^{2,\beta}_{\gamma-2}}$ provided that ε is chosen sufficiently small. One can thus construct the right inverse that satisfies the desired bound.   □

Proposition 38 *Suppose $n = 3$ and choose $\gamma \in (1, 2)$. Then the operator $L_{U,\zeta} : X \to Z$ possesses a right inverse $\mathcal{R} : Z \to X$ satisfying the bound $|\mathcal{R}(f)|_{C^{4,\beta}_{\gamma}} \leq C|f|_{C^{0,\beta}_{\gamma-4}}$ where C is a constant independent of ε.*

Proof The strategy of this proof is similar to that of the previous proof. In fact, the proof of the existence of the right inverse of $\Delta_{U,\zeta} : C^{2,\beta}_{\gamma-2}(\tilde{\Lambda}_{U,\zeta}) \to C^{0,\beta}_{\gamma-4}(\tilde{\Lambda}_{U,\zeta})$, bounded above by a constant independent of ε, still follows from an argument by contradiction as before because $\gamma - 2 \in (-1, 0)$ is in the correct range. What is different is that now the construction of the right inverse for $\Delta_{U,\zeta} + 2(n+1) : C^{4,\beta}_{\gamma}(\tilde{\Lambda}_{U,\zeta}) \to C^{2,\beta}_{\gamma-2}(\tilde{\Lambda}_{U,\zeta})$ has to be modified to take into account the range $\gamma \in (1, 2)$. Indeed, Eq. (34) must be modified by expanding u_{ext} up to order two and cancelling the linear term in the expansion with the linearly growing functions in the kernel of $\Delta_S + 2(n+1)$ on S_0 given in the lists (31) or (32). (One can check that under the symmetry conditions on u_{ext} there are enough such functions to accomplish this.) Then the solution procedure on the neck must be modified by enlarging the deficiency space to include the functions with linear growth on the neck as well. By Lemma 36 one can add a linear combination of these functions to eliminate the linearly growing term in the solution on the neck. This leads to Eq. (35) but with $O(|\lambda|^{-1})$ terms. The remainder of the construction is unchanged.   □

The final estimate in the $n = 2$ case is more complicated still. This is because there is no range of weights for which the Laplacian on punctured manifolds is bijective when $n = 2$ so that one can no longer use a contradiction argument to derive the existence of an appropriately bounded right inverse for $\Delta_{U,\zeta} : [C^{2,\beta}_{\gamma-2}(\tilde{\Lambda}_{U,\zeta})]_{\text{sym}} \to Z$. The patching technique of the previous two propositions must thus be used for $\Delta_{U,\zeta}$ as well as for $\Delta_{U,\zeta} + 2(n+1)$. And both of these tasks are complicated by the fact that the odd harmonic function depending only on the neck coordinate λ has logarithmic growth. This fact makes it necessary to exploit the symmetries of $\tilde{\Lambda}_{U,\zeta}$ even further.

Proposition 39 *Suppose $n = 2$ and choose $\gamma \in (1, 2)$. Then the operator $L_{U,\zeta} : X \to Z$ possesses a right inverse $\mathcal{R} : Z \to X$ satisfying the bound $|\mathcal{R}(f)|_{C^{4,\beta}_{\gamma}} \leq C\varepsilon^{\gamma-2}|f|_{C^{0,\beta}_{\gamma-4}}$ where C is a constant independent of ε.*

Proof The proof has two parts: the construction of a right inverse for $\Delta_{U,\zeta} : [C^{2,\beta}_{\delta}(\tilde{\Lambda}_{U,\zeta})]_{\text{sym}} \to [C^{0,\beta}_{\delta-2}(\tilde{\Lambda}_{U,\zeta})]_{\text{sym}}$ with $\delta \in (-1, 0)$ and one for $\Delta_{U,\zeta} + 6 : [C^{4,\beta}_{\gamma}(\tilde{\Lambda}_{U,\zeta})]_{\text{sym}} \to [C^{2,\beta}_{\gamma-2}(\tilde{\Lambda}_{U,\zeta})]_{\text{sym}}$ with $\gamma \in (1, 2)$. Begin with the first of these constructions, in which the equation $\Delta_{U,\zeta}(w) = f$ for $f \in [C^{0,\beta}_{\delta-2}(\tilde{\Lambda}_{U,\zeta})]_{\text{sym}}$ will be solved according to the following four steps. The condition $\int_{\tilde{\Lambda}_{U,\zeta}} f = 0$ which is built into the space $[C^{0,\beta}_{\delta-2}(\tilde{\Lambda}_{U,\zeta})]_{\text{sym}}$ will be used in a critical way below.

Step 0: Re-balancing the mass of f It will first be shown that one can find a solution of the equation $\Delta_{U,\zeta}(w) = \bar{f}_{\text{trans}}$ where $\bar{f}_{\text{trans}}$ is some function that coincides with f in an annular region around each of the gluing points. Then one will be able to complete the construction of the right inverse by finding local solutions on the neck and on the exterior regions of the equation $\Delta_{U,\zeta}(w) = f - \bar{f}_{\text{trans}}$, which has the advantage that $f - \bar{f}_{\text{trans}}$ vanishes in the

annular region. The usefulness of this will become apparent in Step 3, where it provides the additional precision required to match the local solutions properly.

To begin, consider the gluing points $p_1, p_2 \in S_0$. For any pair of radii $0 < r_0 < r_4$ let $\chi_{\text{trans},r_0,r_4}$ be a smooth, monotone cut-off function that equals one in $\text{Pert}_\varepsilon(S_0) \cap \left[\bigcup_k B_{r_4}(p_k) \backslash B_{r_0}(p_k) \right]$ and transitions to zero in $\text{Pert}_\varepsilon(S_0) \cap \left[\bigcup_k B_{2r_4}(p_k) \backslash B_{r_0/2}(p_k) \right]$. Fix two such radii and define

$$f_{\text{trans}} := f \chi_{\text{trans},r_0,r_4}.$$

An approximate solution of the equation $\Delta_{U,\zeta}(u) = f_{\text{trans}}$ can be found as follows. View f_{trans} as a symmetric function defined on S_0^* carrying a small perturbation of the standard metric, and consider the equation $\Delta_S(u) = f_{\text{trans}}$ on S_0^*. By the theory of the Laplacian on punctured manifolds, $\delta \in (-1, 0)$ is in the range where $\Delta_S : C_\delta^{2,\beta}(S_0^*) \to C_{\delta-2}^{0,\beta}(S_0^*)$ is surjective. Let $w_{\text{trans}} \in C_\delta^{2,\beta}(S_0^*)$ be a solution of this equation and extend it to all of $\tilde{\Lambda}_{U,\zeta}$ by setting

$$\bar{w}_{\text{trans}} := \chi_{\text{trans},r_0/4,4r_4} w_{\text{trans}}$$

and extending by symmetry. Now compute

$$\Delta_{U,\zeta}(\bar{w}_{\text{trans}}) = (\Delta_{U,\zeta} - \Delta_S)(\bar{w}_{\text{trans}}) + [\Delta_S, \chi_{\text{trans},r_0/4,4r_4}](w_{\text{trans}}) + f_{\text{trans}}. \qquad (38)$$

The following estimates are valid. First,

$$|\bar{w}_{\text{trans}}|_{C_\delta^{2,\beta}} \leq C|f|_{C_{\delta-2}^{0,\beta}} \qquad \text{and} \qquad |[\Delta_S, \chi_{\text{trans},r_0/4,4r_4}](w_{\text{trans}})|_{C_{\delta-2}^{0,\beta}} \leq C|f|_{C_{\delta-2}^{0,\beta}}$$

for a constant C independent of ε; and as usual, the first term in (38) is small, satisfying

$$(\Delta_{U,\zeta} - \Delta_S)(\bar{w}_{\text{trans}})|_{C_{\delta-2}^{0,\beta}} \leq C\left(\frac{\varepsilon^{2n}}{r_0^{2n}} + r_4^2 \right) |w_{\text{trans}}|_{C_\delta^{2,\beta}} \leq \frac{1}{2}|f|_{C_{\delta-2}^{0,\beta}}$$

provided ε and r_4 are small enough and $r_0 \geq \kappa\varepsilon$ for κ sufficiently large. This follows from Proposition 17. As a result, the procedure above can be iterated to yield a solution of the equation

$$\Delta_{U,\zeta}(\bar{w}_{\text{trans}}) = \bar{f}_{\text{trans}}$$

where $\bar{f}_{\text{trans}} := f_{\text{trans}} + E_{\text{trans}}$ and E_{trans} has support outside of $\text{Pert}_\varepsilon(S_0) \cap \left[\bigcup_k B_{r_4}(p_k) \backslash B_{r_0}(p_k) \right]$. One has the estimates $|\bar{w}_{\text{trans}}|_{C_\delta^{2,\beta}} + |E_{\text{trans}}|_{C_{\delta-2}^{0,\beta}} \leq C|f|_{C_{\delta-2}^{0,\beta}}$ for a constant C independent of ε. Note that it is necessarily the case that $\int_{\tilde{\Lambda}_{U,\zeta}} \bar{f}_{\text{trans}} = 0$.

Step 1: The neck regions The outcome of Step 0 is that it is now only necessary to solve the equation $\Delta_{U,\zeta}(u) = \hat{f}$ where $\hat{f} := f - \bar{f}_{\text{trans}}$ vanishes in $\text{Pert}_\varepsilon(S_0) \cap \left[\bigcup_k B_{r_4}(p_k) \backslash B_{r_0}(p_k) \right]$ and satisfies $|\hat{f}|_{C_{\delta-2}^{0,\beta}} \leq C|f|_{C_{\delta-2}^{0,\beta}}$. Note that it is necessarily the case that $\int_{\tilde{\Lambda}_{U,\zeta}} \hat{f} = 0$.

Choose $r_1 < r_2 < r_3 \in (r_0, r_4)$ and set $f_{\text{neck}} := \chi_{\text{neck},r_1} \hat{f}$. An approximate solution of the equation $\Delta_{U,\zeta}(w) = f_{\text{neck}}$ will now be found. To this end, view f_{neck} as a function of compact support on the ε-scaled neck N_ε and look for a solution of the equation $\Delta_N(w) = f_{\text{neck}}$ in $C_{\delta'}^{2,\beta}(N_\varepsilon)$ where $\delta' \in (-1, \delta)$. The purpose is to use the compactness of the support of f_{neck} to squeeze some extra decay at the ends of the neck out of the solution. The decomposition results used in Proposition 37 are valid and by using the symmetry of f_{neck}, there is a solution of the form

$$w_{\text{neck}} := v_{\text{neck}} + a \chi_{\text{ext},r_1} L$$

where $v_{\text{neck}} \in C_{\delta'}^{2,\beta}(N_\varepsilon)$ satisfies the estimate $|v_{\text{neck}}|_{C_{\delta'}^{2,\beta}(N_\varepsilon)} \leq Cr_1^{\delta-\delta'}|f|_{C_{\delta-2}^{0,\beta}}$ and $L(\lambda) := \log(|\lambda|) + \tilde{L}(\lambda)$ belongs to the kernel of Δ_N, with $\tilde{L}(\lambda) = O(|\lambda|^{-1})$. The coefficient a can be found explicitly:

$$a = \frac{1}{4\pi} \int f_{\text{neck}} \mathrm{d}\,\mathrm{Vol}_N$$

where dVol_N is the volume form of the metric on N_ε. Finally, extend this solution to all of $\tilde{\Lambda}_{U,\zeta}$ by defining $\bar{w}_{\text{neck}} := \chi_{\text{neck},r_2} w_{\text{neck}}$ and symmetrizing. One has $|\bar{w}_{\text{neck}}|_{C_\delta^{2,\beta}} \leq C\left(r_1^{\delta-\delta'} + \varepsilon^\delta\right)|f_{\text{neck}}|_{C_{\delta-2}^{2,\beta}}$ where ε^δ comes from the estimate $|a| \leq C\varepsilon^\delta|f|_{C_{\delta-2}^{0,\beta}}$.

Step 2: The exterior regions Set $f_{\text{ext}} := \chi_{\text{ext},r_3} \hat{f}$. An approximate solution of the equation $\Delta_{U,\zeta}(w) = f_{\text{ext}}$ will now be found. To this end, view f_{ext} as a function of compact support on S_0^* and look for a solution of the equation $\Delta_S(w) = f_{\text{neck}}$ in $C_{-\delta}^{2,\beta}(S_0^*)$. The decomposition results used in Proposition 37 are valid and by using the symmetry of f_{neck}, there is a solution of the form

$$w_{\text{ext}} := v_{\text{ext}} + a'\chi_{\text{neck},r_3}L'$$

where $v_{\text{neck}} \in C_{-\delta}^{2,\beta}(S_0^*)$ satisfies the estimate $|v_{\text{neck}}|_{C_{-\delta}^{2,\beta}(S_0)} \leq Cr_3^{2\delta}|f|_{C_{\delta-2}^{0,\beta}}$ and L' has logarithmic growth near p_1 and p_2. Without loss of generality, one can assume that $L'(\lambda) := \log(|\lambda|) + \tilde{L}'(\lambda)$ with $\tilde{L}'(\lambda) = O(|\lambda|^{-1})$ in the common λ-coordinate on the region of overlap between the neck and $\mathrm{Pert}_\varepsilon(S_0)$. The coefficient a' can again be found explicitly:

$$a' = -\frac{1}{4\pi} \int f_{\text{ext}} \mathrm{d}\,\mathrm{Vol}_S$$

where Vol_S is the volume form of the standard metric on S_0. Finally, extend this solution to all of $\tilde{\Lambda}_{U,\zeta}$ by defining $\bar{w}_{\text{ext}} := \chi_{\text{ext},r_2} w_{\text{ext}}$ and symmetrizing. One has $|\bar{w}_{\text{ext}}|_{C_\delta^{2,\beta}} \leq C\left(r_3^{2\delta} + \varepsilon^\delta\right)|f_{\text{ext}}|_{C_{\delta-2}^{2,\beta}}$.

Step 3: Estimates and convergence Set $\bar{w} := \bar{w}_{\text{neck}} + \bar{w}_{\text{ext}}$ and consider $\bar{w}$ as the approximate solution of the equation $\Delta_{U,\zeta}(\bar{w}) = \hat{f}$. Before justifying this, it is straightforward to show that

$$a = \int_{\tilde{\Lambda}_{U,\zeta}} f_{\text{neck}} \mathrm{d}\,\mathrm{Vol} + \int_{\tilde{\Lambda}_{U,\zeta}} f_{\text{neck}}(\mathrm{dVol}_N - \mathrm{dVol}) = A + O(r_0^{2+\delta})$$

$$a' = -\int_{\tilde{\Lambda}_{U,\zeta}} f_{\text{ext}} \mathrm{d}\mathrm{Vol} - \int_{\tilde{\Lambda}_{U,\zeta}} f_{\text{ext}}(\mathrm{dVol}_S - \mathrm{dVol}) = A + O(\varepsilon^2)$$

where $A := \int_{\tilde{\Lambda}_{U,\zeta}} f_{\text{neck}}$ and $\int_{\tilde{\Lambda}_{U,\zeta}} f_{\text{ext}} = -A$. Now perform the computation

$$\Delta_{U,\zeta}(\bar{w}) - \hat{f} = \left(\Delta_{U,\zeta} - \Delta_N\right)(\bar{w}_{\text{neck}}) + \left(\Delta_{U,\zeta} - \Delta_S\right)(\bar{w}_{\text{ext}})$$
$$+ [\chi_{\text{neck},r_2}, \Delta_N](w_{\text{neck}}) + [\chi_{\text{ext},r_2}, \Delta_S](w_{\text{ext}})$$
$$+ \chi_{\text{neck},r_2} f_{\text{neck}} + \chi_{\text{ext},r_2} f_{\text{ext}} - \hat{f}$$

$$
= \left(\Delta_{U,\zeta} - \Delta_N \right) (\bar{w}_{\text{neck}}) + \left(\Delta_{U,\zeta} - \Delta_S \right) (\bar{w}_{\text{ext}})
$$
$$
+ [\chi_{\text{neck},r_2}, \Delta_N](v_{\text{neck}}) - [\chi_{\text{neck},r_2}, \Delta_S](v_{\text{ext}})
$$
$$
+ [\chi_{\text{neck},r_2}, \Delta_N](a\tilde{L}) - [\chi_{\text{neck},r_2}, \Delta_S](a'\tilde{L}')
$$
$$
+ [\chi_{\text{neck},r_2}, \Delta_N - \Delta_S](a \log(|\lambda|)) - [\chi_{\text{neck},r_2}, \Delta_S]((a' - a) \log(|\lambda|))
$$
$$
\tag{39}
$$

using the formulæ for a and a' as well as the fact that $\hat{f}$ vanishes between radii r_1 and r_3. The usual analysis can now be invoked to show that each term in (39) is small a small enough multiple of $|f|_{C^{0,\beta}_{\delta-2}}$ in the $C^{0,\beta}_{\delta-2}$ norm by suitable a priori choice of r_i and small enough ε.

The conclusion to be drawn from the work above is that iteration produces a solution of the equation $\Delta_{U,\zeta}(\bar{w}) = \hat{f}$ satisfying the estimate $|\bar{w}|_{C^{2,\beta}_{\delta}} \le C\varepsilon^{\delta}|f|_{C^{0,\beta}_{\delta-2}}$ where C is a constant independent of ε. Coupled with the result of Step 0, one now has a right inverse $\mathcal{R}_1$: $[C^{0,\beta}_{\delta-2}(\tilde{\Lambda}_{U,\zeta})]_{\text{sym}} \to [C^{2,\beta}_{\delta}(\tilde{\Lambda}_{U,\zeta})]_{\text{sym}}$ satisfying the bound $|\mathcal{R}_1(f)|_{C^{2,\beta}_{\delta}} \le C\varepsilon^{\delta}|f|_{C^{0,\beta}_{\delta-2}}$.

It remains to construct the right inverse for $\Delta_{U,\zeta} + 6 : [C^{4,\beta}_{\gamma}(\tilde{\Lambda}_{U,\zeta})]_{\text{sym}} \to [C^{2,\beta}_{\gamma-2}(\tilde{\Lambda}_{U,\zeta})]_{\text{sym}}$ with $\gamma \in (1, 2)$. This task is simpler given the range for γ. In fact, the procedure of Proposition 38 can be used almost verbatim, except replacing w_* with $w_*^{\perp}$ where $*$ refers to either neck or ext and $\perp$ refers to the L^2-projection perpendicular to the constant functions. Thus logarithmic terms will not appear in the solutions u_*. The discrepancy $|w_* - w_*^{\perp}|_{C^{2,\beta}_{\gamma-2}}$ is sufficiently small not to spoil the convergence of the iteration. The result of this analysis is a right inverse $\mathcal{R}_2$: $[C^{2,\beta}_{\gamma-2}(\tilde{\Lambda}_{U,\zeta})]_{\text{sym}} \to [C^{4,\beta}_{\gamma}(\tilde{\Lambda}_{U,\zeta})]_{\text{sym}}$ satisfying the bound $|\mathcal{R}_2(w)|_{C^{4,\beta}_{\gamma}} \le C|w|_{C^{2,\beta}_{\gamma-2}}$. $\qquad\square$

7 The non-linear analysis

7.1 The Non-linear estimates

The remainder of the proof of the Main Theorem is devoted to establishing the final two fundamental estimates (25) and (26) needed to invoke the inverse function theorem. Choose $\gamma \in (4 - n, 0)$ for $n \ge 5$ as well as $\gamma \in (4 - n, 5 - n)$ for $n = 4, 3$ and $\gamma \in (1, 2)$ for $n = 2$.

Proposition 40 *The approximate solution $\tilde{\Lambda}_{U,\zeta}$ satisfies the following estimate. There exists some constant C independent of ε so that*

$$
\|\Phi_{U,\zeta}(0)\|_{C^{0,\beta}_{\gamma-4}(\tilde{\Lambda}_{U,\zeta})} \le Cr_{\varepsilon}^{4-\gamma}.
\tag{40}
$$

Proof By Proposition 17 the divergence of the mean curvature in the exterior region Λ' satisfies

$$
[\rho_{\varepsilon}(x)]^{4-\gamma}|\nabla \cdot H(x)| \le \begin{cases} C\varepsilon^{3n} & x \in \Lambda'' \\ \dfrac{C\varepsilon^{3n}}{r^{3n-2+\gamma}} & \text{dist}(x, p_i) = r \quad \text{and} \quad r \ge r_{\varepsilon}. \end{cases}
$$

By Proposition 25 the divergence of the mean curvature in the neck region $\mathcal{N}$ satisfies

$$
\varepsilon^{4-\gamma}(1+\lambda^2)^{(4-\gamma)/2}|\nabla \cdot H(\lambda, \mu)|
$$
$$
\le C\varepsilon^{4-\gamma}\left[\frac{1}{(1+\lambda^2)^{(n-4+\gamma)/2}} + (1+\lambda^2)^2 \right] \quad \text{for } |\lambda| \le r_{\varepsilon}/\varepsilon.
$$

By Proposition 27 the divergence of the mean curvature in the transition region $\mathcal{T}$ satisfies

$$r_\varepsilon^{4-\gamma} |\nabla \cdot H| \leq C \left[\frac{\varepsilon^{n+1}}{r_\varepsilon^{n-3+\gamma}} + r_\varepsilon^{4-\gamma} \right]$$

The desired supremum estimate is obtained by taking the supremum above and using the ranges for γ and the fact that $\varepsilon < r_\varepsilon$. The estimate of the Hölder coefficient follows similarly.
□

Proposition 41 *The linearization of the contact-stationary Legendrian operator near $\tilde{\Lambda}_{U,\zeta}$ satisfies the following estimate. If ε is sufficiently small, then there is a constant C independent of ε so that*

$$\left\| D\Phi_{U,\zeta}(f)(u) - \mathcal{L}_{U,\zeta}(u) \right\|_{C^{0,\beta}_{\gamma-4}} \leq C\varepsilon^{-2+\gamma} \|f\|_{C^{4,\beta}_\gamma} \|u\|_{C^{4,\beta}_\gamma} \tag{41}$$

for all $u, f \in C^{4,\beta}_\gamma(\tilde{\Lambda}_{U,\zeta})$.

Proof The desired estimate will be proved using scaling arguments as in [3,19]. By compactness, the estimate is certainly true in the region Λ' of $\tilde{\Lambda}_{U,\zeta}$. So consider a small annular region $\Lambda_\sigma := A_\sigma \cap (\mathcal{N} \cup \mathcal{T})$ in $\tilde{\Lambda}_{U,\zeta}$. It is sufficient to perform the calculations for the Lagrangian projection $[\Lambda_\sigma]$ in $\mathbb{CP}^n$. Consider the operator

$$\Phi_\sigma(f) := \nabla \cdot \left(H_{\hat{\phi}_{E_\sigma(f)}[\Lambda_\sigma]} \lrcorner \, \omega_0 \right) \tag{42}$$

where $E_\sigma(f)$ is the extension of a function $f : \Lambda_\sigma \to \mathbb{R}$ to a tubular neighbourhood of $[\Lambda_\sigma]$ and $\hat{\phi}_{E_\sigma(f)}$ is the associated time-one Hamiltonian flow.

The next step is to determine how all the objects in (42) scale with σ. First, $[\Lambda_\sigma] = \sigma[\Lambda_1]$ where $\Lambda_1 = A_1 \cap \sigma^{-1}(\mathcal{N} \cup \mathcal{T})$. Moreover, $[\Lambda_1]$ is some Lagrangian submanifold whose geometry is bounded by a universal constant since the norm of the second fundamental form of $[\Lambda_\sigma]$ in the annulus A_σ is $O(\sigma^{-1})$. Suppose that $[\Lambda_1]$ carries the metric g_1 so that $[\Lambda_\sigma]$ carries the metric $g_\sigma = \sigma^2 g_1$. Since $E_\sigma(f) = f \circ \exp^{-1}_{\Lambda_\sigma}$ near Λ_σ, then one can check that $\hat{\phi}_{E_\sigma(f)}([\Lambda_\sigma]) = \sigma\hat{\phi}_{E_1(f/\sigma^2)}([\Lambda_1])$. Consequently,

$$\Phi_\sigma(f) = \nabla \cdot \left(H \left(\sigma\hat{\phi}_{E_1(f/\sigma^2)}([\Lambda_1]) \right) \lrcorner \, \omega_0 \right) = \frac{1}{\sigma^2} \Phi_1 \left(\frac{f}{\sigma^2} \right) \tag{43}$$

using the scaling property of the mean curvature and the covariant derivative under conformal transformation. All quantities on the right hand side of (43) refer to the metric g_1.

One can now derive the desired estimate. Equation (43) implies that $D\Phi_\sigma$ in A_σ must satisfy

$$\left| D\Phi_\sigma(f)(u) - D\Phi_\sigma(0)(u) \right| = \frac{1}{\sigma^2} \left| D\Phi_1 \left(\frac{f}{\sigma^2} \right) \left(\frac{u}{\sigma^2} \right) - D\Phi_1(0) \left(\frac{u}{\sigma^2} \right) \right|$$

$$\leq \frac{C}{\sigma^6} \|f\|_{C^4(\Lambda_1)} \|u\|_{C^4(\Lambda_1)} \tag{44}$$

where C is some universal constant pertaining to Φ_1 on Λ_1. Multiplying by $\sigma^{4-\gamma}$ and reversing the scaling in Eq. (44) gives

$$\left\| D\Phi_\sigma(f)(u) - D\Phi_\sigma(0)(u) \right\|_{C^4_{\gamma-4}(\Lambda_\sigma)} \leq C\sigma^{-2+\gamma} \|f\|_{C^4_\gamma(\Lambda_\sigma)} \|u\|_{C^4_\gamma(\Lambda_\sigma)}, \tag{45}$$

where all quantities in (45) refer to the metric g_σ. One can now piece the above estimates together for different A_σ, using the fact that the smallest σ can be is $O(\varepsilon)$ in the center

of the neck region, and obtain the desired supremum estimate. The estimate of the Hölder coefficient follows similarly.   □

7.2 The proof of the Main Theorem

The Main Theorem to be proved in this paper, re-formulated in the more technical language of the preceding sections, is as follows.

Theorem 42 *Let $U \in SU(n+1)$ be as in Example* 1 *or* 2 *and let $\zeta = e^{2\pi i a/kN}$ be such that it is possible to construct an approximate solution $\tilde{\Lambda}_{U,\zeta}$ as in Sect.* 4. *Let X, Z be the Banach subspace of functions on $\tilde{\Lambda}_{U,\zeta}$ defined in Definition* 31. *Choose $\gamma \in (4 - n, 0)$ for $n \geq 5$ as well as $\gamma \in (4 - n, 5 - n)$ for $n = 4, 3$ and $\gamma \in (1, 2)$ for $n = 2$. Set $r_\varepsilon := \varepsilon^s$. Then there is $s \in (0, 1)$ so that a solution of the contact-stationary Legendrian equation $\Phi_{U,\zeta}(f) = 0$ can be found with $f \in X$ satisfying the bound $|f|_{C_\gamma^{4,\beta}} \leq C r_\varepsilon^{4-\gamma}$.*

Proof The proof is a re-organization and summary of all the main results from the previous sections. It is necessary to establish the following facts. First, one must show that $\limsup_\varepsilon C(\varepsilon) \| Q_{U,\zeta} \|_{\mathrm{op}} < 1/2$ where $\| Q_{U,\zeta} \|_{\mathrm{op}}$ is the operator norm of $Q_{U,\zeta}$ calculated in Lemma 33 and $C(\varepsilon)$ is the upper bound for the right inverse of $\mathcal{L}_{U,\zeta}$ found in Sect. 6.2. Since $|\Phi_{U,\zeta}(0)|_{C_{\gamma-4}^{0,\beta}} = O(r_\varepsilon^{4-\gamma})$ found in Proposition 40 and $R(\varepsilon) = O(\varepsilon^{2-\gamma})$ found in Proposition 41, one must then show that $r_\varepsilon^{4-\gamma} \varepsilon^{\gamma-2} C(\varepsilon)$ can be made as small as desired, where $C(\varepsilon)$ is the upper bound for the right inverse of $\mathcal{L}_{U,\zeta}$ found in Sect. 6.2.

- When $n \geq 3$ then $C(\varepsilon) = O(1)$ and
$$\lim_{\varepsilon \to 0} \| Q_{U,\zeta} \| \leq C \cdot \lim_{\varepsilon \to 0} \left(\varepsilon^{n-s(n-2)} + \varepsilon^2 + \varepsilon^{4s} \right) = 0$$
  provided $s < n/(n-2)$. Also, $r_\varepsilon^{4-\gamma} \varepsilon^{\gamma-2} = \varepsilon^{(4-\gamma)s+\gamma-2} \to 0$ when $s > (2-\gamma)/(4-\gamma)$. By choice of γ and n, the range of such $s \in (0, 1)$ is non-empty and thus the theorem is true.

- When $n = 2$ then $C(\varepsilon) = O(\varepsilon^{\gamma-2})$ and
$$\lim_{\varepsilon \to 0} C(\varepsilon) \| Q_{U,\zeta} \| \leq C \cdot \lim_{\varepsilon \to 0} \left(\varepsilon^2 + \varepsilon^{4s} \right) \varepsilon^{\gamma-2} = 0$$
  provided $s > (2-\gamma)/4$. Also, $r_\varepsilon^{4-\gamma} \varepsilon^{2\gamma-4} = \varepsilon^{(4-\gamma)s+2\gamma-4} \to 0$ when $s > (4-2\gamma)/(4-\gamma)$. By choice of γ the range of such $s \in (0, 1)$ is non-empty and thus the theorem is true.

This completes the proof of the theorem.   □

7.3 Embeddedness of the solutions

Denote the solution of the contact-stationary Legendrian problem constructed from $U \in SU(n+1)$ with $\zeta = e^{2\pi i/k}$ in the previous sections by f_k. Let ϕ_{f_k} be the contact deformation constructed from f_k using the method developed in Sect. 5.3. A simple chain of reasoning shows that the deformed submanifold $\phi_{f_k}(\tilde{\Lambda}_{U,\zeta})$ is embedded whenever $\tilde{\Lambda}_{U,\zeta}$ itself is, so long as k is sufficiently large.

Suppose that $\tilde{\Lambda}_{U,\zeta}$ is embedded in $\mathbb{S}^{2n+1}$. Then $\tilde{\Lambda}_{U,\zeta}$ is contained in some non-self-intersecting tubular neighbourhood of itself. The width of this tubular neighbourhood is clearly larger in the parts of $\tilde{\Lambda}_{U,\zeta}$ that are subsets of some $U^s(S_0)$ and smaller in the neck regions of $\tilde{\Lambda}_{U,\zeta}$. In fact, one can argue based on scaling that the width of the tubular

neighbourhood at a point $p \in \tilde{\Lambda}_{U,\zeta}$ is $O(\rho_\varepsilon(p))$ where ρ_ε is the weight function from Definition 28 that is used to define the $C_\gamma^{l,\beta}$ norm. The question of embeddedness can now be re-phrased in terms of this tubular neighbourhood: $\phi_{f_k}(\tilde{\Lambda}_{U,\zeta})$ fails to be embedded if either $\phi_{f_k}(\tilde{\Lambda}_{U,\zeta})$ has local self-intersection somewhere within the tubular neighbourhood, or else $\phi_{f_k}(\tilde{\Lambda}_{U,\zeta})$ intersects itself by leaving the tubular neighbourhood somewhere and re-entering it somewhere else.

In order to decide if $\phi_{f_k}(\tilde{\Lambda}_{U,\zeta})$ intersects itself in one of these two ways, one must understand how 'far' the contact deformation ϕ_{f_k} can move the points of $\tilde{\Lambda}_{U,\zeta}$. Recall that ϕ_{f_k} is the time-one flow of the contact vector field corresponding to the function $E(f_k)$ that is an extension of f_k orthogonal to $\tilde{\Lambda}_{U,\zeta}$. Thus the distance of $\phi_{f_k}(p)$ from p is governed by the size of this vector field, which in turn is governed by the size of f_k (in the Hopf direction) and the first derivative of f_k (in contact directions) so long as these quantities are sufficiently small. Moreover, $\phi_{f_k}(\tilde{\Lambda}_{U,\zeta})$ remains graphical over $\tilde{\Lambda}_{U,\zeta}$ so long as the derivative of ϕ_{f_k} remains sufficiently small, which in turn requires that f_k is small up to its second derivative. In fact, it is necessary to have $|f_k(x)| + |\nabla f_k(x)| = O(\rho_\varepsilon(x))$ and $|\nabla^2 f_k(x)| \le O(1)$ for all $x \in \tilde{\Lambda}_{U,\zeta}$. It is now a simple matter to verify that the estimates of the size of f_k and its derivatives from Theorem 42 imply that the requirements for embeddedness are met, provided the approximate solution $\tilde{\Lambda}_{U,\zeta}$ is an embedded submanifold of $\mathbb{S}^{2n+1}$ to begin with.

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