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Constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$ by gluing spherical building blocks

Adrian Butscher

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Abstract The techniques developed by Butscher (Gluing constructions amongst constant mean curvature hypersurfaces of $\mathbb{S}^{n+1}$) for constructing constant mean curvature (CMC) hypersurfaces in $\mathbb{S}^{n+1}$ by gluing together spherical building blocks are generalized to handle less symmetric initial configurations. The outcome is that the approximately CMC hypersurface obtained by gluing the initial configuration together can be perturbed into an exactly CMC hypersurface only when certain global geometric conditions are met. These *balancing conditions* are analogous to those that must be satisfied in the "classical" context of gluing constructions of CMC hypersurfaces in Euclidean space, although they are more restrictive in the $\mathbb{S}^{n+1}$ case. An example of an initial configuration is given which demonstrates this fact; and another example of an initial configuration is given which possesses no symmetries at all.

1 Introduction

Gluing constructions of constant mean curvature hypersurfaces. A constant mean curvature (CMC) hypersurface Λ contained in an ambient Riemannian manifold M of dimension $n+1$ has the property that its mean curvature with respect to the induced metric is constant. This property ensures that the n-dimensional area of Λ is a critical value of the area functional for hypersurfaces of M subject to an enclosed-volume constraint. One very important method for constructing CMC hypersurfaces is the *gluing technique* in which a more complex CMC hypersurface is built up from simple CMC building blocks. This technique was pioneered by Kapouleas in the context of CMC hypersurfaces in $\mathbb{R}^3$ [6–8]. The idea is that a very good approximation of a CMC hypersurface can be constructed by forming the connected sum of an initial configuration of simple CMC building blocks, which can then be perturbed to an exactly CMC hypersurface if certain global geometric conditions, called *balancing conditions*, are satisfied by the initial configuration.

A. Butscher (✉)
Department of Mathematics, Stanford University, Stanford, CA 94305, USA
e-mail: butscher@stanford.edu

The gluing technique has been a very successful method for constructing CMC hypersurfaces in $\mathbb{R}^3$, with the proviso that the resulting hypersurfaces are always small perturbations of the simple building blocks from which they are constructed, namely spheres and nearly singular truncated Delaunay surfaces. This is because the quality of the approximate solution that one can construct improves as the approximate solution more and more closely resembles a union of mutually tangent spheres. Although it is easy to imagine how to use the gluing technique in ambient manifolds other than $\mathbb{R}^3$, provided one has enough simple building blocks, it is not clear that the gluing technique will be quite as successful, in particular when the ambient manifold is compact.

In Butscher's and Butscher-Pacard's work [2–4], the gluing technique for constructing CMC hypersurfaces has been successfully adapted to work in the compact ambient manifold $\mathbb{S}^{n+1}$. In these papers, the CMC building blocks of the sphere—namely the hyperspheres obtained by intersecting $\mathbb{S}^{n+1}$ with hyperplanes and the product spheres of the form $\mathbb{S}^p(\cos(\alpha)) \times \mathbb{S}^q(\sin(\alpha))$ for $\alpha \in (0, \pi/2)$ called the generalized Clifford tori—are configured in a variety of different ways, glued together using small embedded catenoidal necks, and perturbed into CMC hypersurfaces. One should imagine that the hypersurfaces constructed in the papers are analogues of "classical" constructions possible in Euclidean space. Indeed, there are obstructions for solving the CMC equation on an arbitrary initial configuration in $\mathbb{S}^{n+1}$ that are analogous to the obstructions appearing in Euclidean space; and just as in the Euclidean setting, when certain global geometric conditions are met, the obstructions disappear. However, the geometric conditions found in [2–4] for $\mathbb{S}^{n+1}$ constructions, though close analogues of the conditions identified by Kapouleas for Euclidean space, appear to be somewhat stronger. This is to be expected since $\mathbb{S}^{n+1}$ is compact and the additional requirement that the initial configurations must close should have ramifications in the analysis of the CMC equation.

The balancing condition is best explained in the more general context found in Korevaar, Kusner and Solomon's work [9]. First, suppose that Λ is a hypersurface with constant mean curvature h in an $(n+1)$-dimensional Riemannian manifold (M, g) possessing a Killing field V. Let $\mathcal{U}$ be an open set in Λ and $\bar{\mathcal{U}}$ be an open set in M such that $\partial\bar{\mathcal{U}} = \partial\mathcal{U} \cup C$ where C is a bounded n-dimensional cap which may have multiple components. Then the first variation formula for the n-volume of $\mathcal{U}$ subject to the constraint of constant enclosed $(n+1)$-volume of $\bar{\mathcal{U}}$ in the direction of the variation determined by V implies

$$\int_{\partial\mathcal{U}} g(\nu, V) - h \int_{C} g(\eta, V) = 0 \tag{1}$$

where ν is the unit normal vector field of $\partial\mathcal{U}$ in Λ and η is the unit normal vector field of C in M. This formula can now applied to the approximate solution of the CMC perturbation problem, having mean curvature approximately equal to h, in the following way. Choose the open set $\mathcal{U}$ as one of the building blocks of the approximate solution. Then $\partial\mathcal{U}$ consists of a disjoint union of small $(n-1)$-spheres at the centres of the necks attaching $\mathcal{U}$ to its neighbours, and C is the disjoint union of the small disks that cap these spheres off. The left hand side of (1) now encodes information about the width and location of the neck regions of $\mathcal{U}$. If one hopes to perturb $\mathcal{U}$ into a piece of CMC hypersurface for which (1) holds exactly, then one should expect to start with $\mathcal{U}$ for which (1) holds in some approximate sense, to be made precise later on. An approximately CMC hypersurface for which this is true is called *balanced*.

The balancing condition amounts to a form of local symmetry satisfied by each building block with respect to its nearest neighbours in the initial configuration that is to be glued together. This is similar to what happens in Euclidean space. However, force balancing in itself is not the end of the story—a balanced approximate solution can not necessarily be perturbed to an exactly CMC hypersurface. It is in addition necessary to be able to re-position the various building blocks with respect to each other so as to maintain the force balancing condition even under small perturbations. Technically speaking, this amounts to the requirement that the mapping taking a re-positioned approximate solution to a set of small real numbers via the integrals on the left hand side of (1) be surjective. This requirement also exists in the Euclidean case, but is more restrictive in the case of $\mathbb{S}^{n+1}$. In fact, only by imposing a high degree of symmetry on their initial configurations are Butscher and Butscher-Pacard able to satisfy both types of obstruction to the solvability of the CMC equation.

One impression that the reader might have, after studying the implementation of the gluing technique in $\mathbb{S}^{n+1}$ presented in Butscher and Butscher-Pacard's papers, is that it might be impossible to construct CMC hypersurfaces in $\mathbb{S}^{n+1}$ without imposing a high degree of symmetry. Indeed, the totality of local symmetry conditions imposed by force balancing and the fact that CMC hypersurfaces in $\mathbb{S}^{n+1}$ must close seems to force a degree of global symmetry on the initial configuration; and the methods developed in [2] do not seem to apply perfectly to initial configurations with small symmetry groups. However, this paper will show that it is quite possible to develop a gluing technique that is applicable to initial configurations with less symmetry.

Statement of results. The theorem that will be proved in this paper can be explained as follows. Let $\Gamma := \{\gamma_1, \ldots, \gamma_L\}$ be a set of oriented geodesic segments with the property that the one-dimensional variety $\bigcup_s \gamma_s$ has no boundary. Without loss of generality: the points of contact between any two segments are always amongst the endpoints of the segments; and two segments are never parallel whenever they meet. Thus the endpoints of each geodesic segment γ_s make contact with at least two other segments. Let $\{p_1, \ldots, p_M\}$ be the set of all endpoints of the geodesic segments and for each p_s let $T_{1,s}, \ldots, T_{N_s,s} \in T_{p_s}\mathbb{S}^{n+1}$ be the unit tangent vectors of the geodesics emanating from p_s. Now position hyperspheres of radius $\cos(\alpha)$ separated by a distance τ_s along each of the geodesics, perhaps winding multiple times around $\mathbb{S}^{n+1}$. Note that there is a transcendental relationship between the τ_s and the number of windings around γ_s that must be satisfied for this to be possible. Denote this initial configuration of hyperspheres by $\Lambda^{\#}_{\Gamma,\tau}$.

In Sect. 2 a procedure will be developed for gluing the hyperspheres in $\Lambda^{\#}_{\Gamma,\tau}$ together by embedding small catenoidal necks between each pair of hyperspheres to form an *approximate solution* $\tilde{\Lambda}_{\Gamma,\tau}$ of the CMC deformation problem. It will be shown in Sect. 4 that $\tilde{\Lambda}_{\Gamma,\tau}$ is balanced if

$$\sum_{j=1}^{N_s} \varepsilon_{j,s}^{n-1} T_{j,s} = 0 \tag{2}$$

for each point p_s, where $\varepsilon_{j,s}$ is a parameter related to the separation parameter $\tau_{j,s}$ along the geodesic whose tangent vector is $T_{j,s}$. (Actually, $\varepsilon_{j,s}$ is the width of the neck connecting the hypersphere at p_s to its neighbour in the direction of $T_{j,s}$. The relation with $\tau_{j,s}$ will be established during the description of the gluing process).

Main Theorem 1 *Let $\Lambda^{\#}_{\Gamma,\tau}$ be the initial configuration of hyperspheres described above. Suppose that balancing condition (2) holds and also that the mapping between finite-dimensional vector spaces which takes small displacements of the geodesics forming $\Lambda^{\#}_{\Gamma,\tau}$ to the quantity given by the left hand side of (2) has full rank. If τ is sufficiently small, then $\tilde{\Lambda}_{\Gamma,\tau}$ can be perturbed into an exactly CMC hypersurface $\Lambda_{\Gamma,\tau}$. This hypersurface can be* *described as a normal graph over $\tilde{\Lambda}_{\Gamma,\tau}$ where the graphing function has small $C^{2,\beta}$-norm. In particular, $\Lambda_{\Gamma,\tau}$ is embedded if and only if $\tilde{\Lambda}_{\Gamma,\tau}$ is embedded.*

The proof of this theorem will follow broadly the same lines as Main Theorem 2 in Butscher's paper [2]. That is, it will be shown that the partial differential equation for the graphing function whose solution gives a CMC perturbation of $\tilde{\Lambda}_{\Gamma,\tau}$ can be solved up to a error term belonging to a finite dimensional obstruction space spanned by the approximate Jacobi fields of $\tilde{\Lambda}_{\Gamma,\tau}$ (as explained more fully in [2] and in the proof below). Then it will be shown that the balancing conditions given in the theorem above are sufficient to eliminate the error term.

2 Construction of the approximate solution

This section of the paper carries out the construction of the approximately CMC submanifolds which will be perturbed into exactly CMC submanifolds. The construction is in most respects identical to the construction carried out in [2] and will thus be sketched somewhat briefly. An important point of departure from [2] should be mentioned: since less symmetric configurations are being considered in this paper, it is necessary to build greater flexibility into the approximate solutions than was needed in [2].

2.1 The initial configuration of hyperspheres

Write $\mathbb{R}^{n+2}$ as $\mathbb{R} \times \mathbb{R}^{n+1}$ and give it the coordinates $(x^0, x^1, \ldots, x^{n+1})$. Consider the hypersphere

$$S_\alpha := \{x \in \mathbb{R}^{n+2} \,:\, x^0 = \cos\alpha \ \text{ and } \ (x^1)^2 + \cdots + (x^{n+1})^2 = \sin^2(\alpha)\}\,.$$

This hypersphere has constant mean curvature H_α. An arbitrary configuration of rotated copies of S_α positioned along geodesic segments can be defined concretely as follows.

First let $\Gamma := \{\gamma_1, \ldots, \gamma_L\}$ be a set of oriented geodesic segments with the property that the one-dimensional variety $\bigcup_s \gamma_s$ has no boundary. Without loss of generality: the points of contact between any two segments are always amongst the endpoints of the geodesics; and two segments are never parallel whenever they meet. Thus the endpoints of each geodesic segment γ_s make contact with at least two other segments. Let $|\gamma_s|$ be the length of γ_s and use $\gamma_s(t)$ to denote the point on γ_s lying a distance t from its starting point. Hence $t \longmapsto \gamma_s(t)$ is the arc length parametrization of γ_s. Suppose that there is one fixed $\alpha \in (0, \pi/2)$ along with positive integers N_s and m_s and small separation parameters $\tau_s > 0$ so that $|\gamma_s| + 2\pi m_s = N_s(2\alpha + \tau_s)$ for each $s = 1, \ldots, L$.

Define the points $\mathring{p}_{sk} := \gamma_s(k(2\alpha + \tau_s))$ as well as the hyperspheres $\mathring{S}^{sk}_\alpha := \partial B_\alpha(\mathring{p}_{sk})$. Thus the $\mathring{S}^{sk}_\alpha$ for $k = 0, \ldots, N_s$ are a collection of N_s hyperspheres of the same mean curvature winding around the geodesic γ_s a number m_s times and separated from each other by a distance τ_s. The proof of the Main Theorem will in addition require small displacements of the hyperspheres above from these "equilibrium" positions. To this end, introduce the small displacement parameters $\vec{\sigma}_{sk} \in T_{\mathring{p}_{sk}}\mathbb{S}^{n+1}$ and define the points $p_{sk} := \exp_{\mathring{p}_{sk}}(\vec{\sigma}_{sk})$ as well as the hyperspheres $S^{sk}_\alpha[\vec{\sigma}_{sk}] := \partial B_\alpha(p_{sk})$. To avoid ambiguity, the displacement parameter for any hypersphere corresponding to an endpoint of a geodesics must be unique; this is achieved by setting the appropriate $\vec{\sigma}_{s0}$ and $\vec{\sigma}_{s' N_{s'}}$ equal. One should note that each hypersphere S^{sk}_α now has at least two nearest neighbours. If $k \neq 0, N_s$ then S^{sk}_α is situated near an interior point of the geodesic γ_s and thus has exactly two nearest neighbours $S^{s,k-1}_\alpha$ and $S^{s,k+1}_\alpha$ along this geodesic. If $k = 0$ or N_s then S^{sk}_α is situated near an endpoint of the geodesic γ_s and has strictly greater than two nearest neighbours corresponding to hyperspheres of the form $S^{s'k'}_\alpha$ where $s' \in \{0, \ldots, L\}\backslash\{s\}$ and $k' = 1$ or $N_{s'} - 1$.

The initial configuration of hyperspheres is defined as follows.

Definition 1 The *initial configuration* of hyperspheres of mean curvature H_α positioned along the collection of geodesics Γ having separation parameters $\tau := \{\tau_1, \ldots, \tau_L\}$ and displacement parameters $\vec{\sigma} := \{\vec{\sigma}_{10}, \ldots, \vec{\sigma}_{LN_L}\}$ is defined to be

$$\Lambda^\#[\alpha, \Gamma, \tau, \vec{\sigma}] := \bigcup_{s=1}^{L} \bigcup_{k=0}^{N_s} S^{sk}_\alpha[\vec{\sigma}_{sk}] .$$

Note that there is redundancy in the labeling above due to the intersections amongst the geodesics.

Finally, one can choose once and for all an $SO(n+2)$-rotation $R_{sk}[\vec{\sigma}_{sk}]$ taking $S^{sk}_\alpha[\vec{\sigma}_{sk}]$ to S_α as follows. First fix a particular $R_{sk} \in SO(n+2)$ take $\mathring{S}^{sk}_\alpha$ to S_α (here, the choice does not matter so long as it is fixed *a priori*). Then let $\mathcal{W}_{\vec{\sigma}_{sk}}$ be the distance-one rotation in the one-parameter family of rotations generated by the $(n+2) \times (n+2)$ anti-symmetric linear transformation given by $W_{\vec{\sigma}_{sk}}(X) := \langle \vec{\sigma}_{sk}, X \rangle \mathring{p}_{sk} - \langle \mathring{p}_{sk}, X \rangle \vec{\sigma}_{sk}$ for $X \in \mathbb{R}^{n+2}$. This is the unique $SO(n+2)$-rotation that coincides with $\exp_{\mathring{p}_{sk}}(\vec{\sigma}_{sk})$ at $\mathring{p}_{sk}$. Now define

$$R_{sk}[\vec{\sigma}_{sk}] := R_{sk} \circ \mathcal{W}^{-1}_{\vec{\sigma}_{sk}} . \tag{3}$$

A consequence of this choice is that the dependence of $R_{sk}[\vec{\sigma}_{sk}]$ on $\vec{\sigma}_{sk}$ is smooth.

2.2 Symmetries

Let G_Γ be the largest subgroup of $O(n+2)$ preserving the collection of geodesics Γ. The idea is that G_Γ should become the group of symmetries of the CMC hypersurface constructed in the proof of the Main Theorem. Therefore in all steps leading up to the proof of the Main Theorem, it will be necessary to ensure that invariance with respect to G_Γ is preserved.

The initial configuration $\Lambda^\#[\alpha, \Gamma, \tau, 0]$ is clearly invariant with respect to G_Γ but once non-zero displacement parameters are introduced, this may no longer be so. To preserve G_Γ-invariance, it will be necessary to choose only special values of the displacement parameters. Let $N := \sum_{s=1}^{L}(N_s + 1)$ be the total number of hyperspheres in $\Lambda^\#[\alpha, \Gamma, \tau, 0]$ so that there are a total of N displacement parameters, each of which belongs to $\mathbb{R}^n$. Define the set

$$\mathcal{D}_\Gamma := \left\{ \vec{\sigma} \in \mathbb{R}^n \times \overset{N \text{ times}}{\cdots\cdots} \times \mathbb{R}^n : \Lambda^\#[\alpha, \Gamma, \tau, \vec{\sigma}] \text{ is } G_\Gamma\text{-invariant} \right\} . \tag{4}$$

Henceforth the condition $\vec{\sigma} \in \mathcal{D}_\Gamma$ on the displacement parameters will be assumed.

2.3 Preliminary perturbation of the initial configuration

As in [2], the first step in gluing the initial configuration $\Lambda^\#[\alpha, \Gamma, \tau, \vec{\sigma}]$ together is to replace each hypersphere $S^{sk}_\alpha[\vec{\sigma}_{sk}]$ in $\Lambda^\#[\alpha, \Gamma, \tau, \vec{\sigma}]$ by a small normal perturbation of itself. The purpose is to give each hypersphere a catenoidal shape near its gluing points.

To proceed, recall that $S^{sk}_\alpha[\vec{\sigma}_{sk}] = R_{sk}[\vec{\sigma}_{sk}]^{-1}(S_\alpha)$. Let $p_1, \ldots, p_K$ be the images under $R_{sk}[\vec{\sigma}_{sk}]$ of the points on $S^{sk}_\alpha[\vec{\sigma}_{sk}]$ that are nearest to the neighbours of $S^{sk}_\alpha[\vec{\sigma}_{sk}]$ amongst the hyperspheres of $\Lambda^{\#}[\alpha, \Gamma, \tau, \vec{\sigma}]$. Introduce a small radius parameter r to be determined later and define

$$\tilde{S}_{\alpha}^{sk}[\vec{a}_{sk}, \vec{\sigma}_{sk}] := (R_{sk}[\vec{\sigma}_{sk}])^{-1} \circ \exp\left(G_{sk} N_{\alpha}\right) \left(S_{\alpha} \setminus \bigcup_{j=1}^{K} B_r(p_j) \right), \tag{5}$$

where $G_{sk} : S_{\alpha} \setminus \{p_1, \ldots, p_K\} \to \mathbb{R}$ is the function determined by the following procedure. Refer to $(a_1, \ldots, a_K)$ as the *asymptotic parameters* of $\tilde{S}_{\alpha}^{sk}[\vec{a}_{sk}, \vec{\sigma}_{sk}]$.

Let $\mathcal{L}_{\mathbb{S}^n} := \Delta_{\mathbb{S}^n} + n$ be the linearized mean curvature operator of S_{α} and recall that the smooth kernel of $\mathcal{L}_{\mathbb{S}^n}$ consists of the linear span of the restrictions of the coordinate functions $q^t := x^t|_{\mathbb{S}^n}$ for $t = 1, \ldots, n+1$. Let $\delta(p_j)$ be the Dirac δ-mass centered at the point p_j. Then for each $\vec{a}_{sk} := (a_1, \ldots a_K) \in \mathbb{R}^K$, one can find a unique solution $G_{sk} : S_{\alpha} \setminus \{p_1, \ldots, p_K\} \to \mathbb{R}$ of the distributional equation

$$\mathcal{L}_{\mathbb{S}^n}(G_{sk}) = \sum_{j=1}^{K} a_j \left(\delta(p_j) - \sum_{t=1}^{n} \lambda_j^t \, \chi \cdot q^t \right) \tag{6}$$

that is L^2-orthogonal to the smooth kernel of $\mathcal{L}_{\mathbb{S}^n}$. Here χ is a cut-off function vanishing in a neighbourhood of each of the p_j that will be defined precisely later, and the $\lambda_j^t \in \mathbb{R}$ are coefficients designed to ensure that the right hand side of (6) is L^2-orthogonal to all the q^t, thereby guaranteeing the existence of the solution.

2.4 Assembling the approximate solution

One must now find the catenoidal necks that fit optimally in the space between the various perturbed hyperspheres produced in the previous section. Then one can construct a smooth approximate solution by interpolating from one perturbed hypersphere to the next through these necks. Here is a summary of how this was done in [2]. The same method works here.

Choose two hyperspheres S and S' in $\Lambda^{\sharp}[\alpha, \Gamma, \tau, \vec{\sigma}]$ positioned somewhere on the geodesic γ along with their associated perturbed hyperspheres $\tilde{S}$ and $\tilde{S}'$. Let $p^{\flat}$ be the midpoint between the centers of S and S' on γ and consider the images in $\mathbb{R}^{n+1}$ of $\tilde{S}$ and $\tilde{S}'$ under the stereographic projection sending $p^{\flat}$ to the origin and γ to the (y^1, y^2)-coordinate axis. Here $\mathbb{R}^{n+1}$ has been given coordinates $(y^1, \ldots, y^{n+1})$. From the formulæ for normal graphs over hyperspheres and the properties of the stereographic projection, one finds that the images of $\tilde{S}$ and $\tilde{S}'$ can be given as graphs over the $\hat{y} := (y^2, \ldots, y^{n+1})$ factor in the form $y^1 := \pm\mathcal{G}^{\pm}(\|\hat{y}\|)$ where the function $\mathcal{G}^{\pm}$ has the expansion

$$\mathcal{G}^{\pm}(\|\hat{y}\|) = \begin{cases} -\tan(\tau/4) - \frac{\|\hat{y}\|^2}{2r} + a^{\pm}\left(c_2 - C_2 \log(\|\hat{y}\|)\right) + \mathcal{O}(\|\hat{y}\|^4) & n = 2 \\ -\tan(\tau/4) - \frac{\|\hat{y}\|^2}{2r} + \frac{a^{\pm} C_3}{\|\hat{y}\|} + \mathcal{O}(\|\hat{y}\|^4) & n \geq 3 \end{cases} \tag{7}$$

near $\hat{y} = 0$. Here c_2, C_2, C_3, C_4 and C_n are constants, while $a^{\pm}$ are the asymptotic parameters corresponding to the gluing points of S and S' under consideration, respectively.

The standard catenoid in $\mathbb{R}^{n+1}$, scaled by a factor $\varepsilon > 0$, can be written as the union of two graphs over the $\hat{y}$-coordinates. That is $\varepsilon\Sigma := \Sigma_{\varepsilon}^{+} \cup \Sigma_{\varepsilon}^{-}$ where $\Sigma_{\varepsilon}^{\pm} := \big\{(\pm\varepsilon F(\|\hat{y}\|/\varepsilon), \hat{y}) : \|\hat{y}\| \geq \varepsilon\big\}$ and the function $F : \{x \in \mathbb{R} : x \geq 1\} \to \mathbb{R}$ is defined by

$$F(x) := \int_1^x (\sigma^{2n-2} - 1)^{-1/2} \mathrm{d}\sigma \,.$$

In dimension $n = 2$ this function is simply $F(x) = \operatorname{arccosh}(x)$. Therefore one has the expansion

$$\varepsilon F(\|\hat{y}\|/\varepsilon) = \begin{cases} \varepsilon \log(2/\varepsilon) + \varepsilon \log(\|\hat{y}\|) - \frac{\varepsilon^3}{4\|\hat{y}\|^2} + \mathcal{O}\left(\frac{\varepsilon^5}{\|\hat{y}\|^4}\right) & n = 2 \\ \varepsilon c_n - \frac{\varepsilon^{n-1}}{(n-2)\|\hat{y}\|^{n-2}} - \frac{\varepsilon^{3n-3}}{2(3n-4)\|\hat{y}\|^{3n-4}} + \mathcal{O}\left(\frac{\varepsilon^{5n-5}}{\|\hat{y}\|^{5n-6}}\right) & n \geq 3 \end{cases} \tag{8}$$

when $\|\hat{y}\|/\varepsilon$ is large, where c_n is yet another constant. In order to find the optimally matching catenoid, one must compare the asymptotic expansions (7) with the asymptotic expansion (8) and choose ε, a^+ and a^- so that the leading-order terms coincide. As in [2], these matching conditions determine $a^\pm$ and ε completely in terms of the separation τ between the hyperspheres. In fact, one has $\varepsilon = c_n^{-1}\tan(\tau/4)$ in dimension $n \geq 3$ and ε satisfies $\tan(\tau/4) = c_2 C_2^{-1}\varepsilon + \varepsilon\log(2/\varepsilon)$ in dimension $n = 2$.

To complete the gluing, one defines a new surface in $\mathbb{R}^{n+1}$ that interpolates between the images of $\tilde{S}$ and $\tilde{S}'$ and the appropriately scaled and truncated catenoid $\varepsilon\Sigma$. This interpolating surface is the union of the two graphs $\tilde{\Sigma}^\pm_\varepsilon = \{(\pm\tilde{F}^\pm_{a,\tau}(\hat{y}), \hat{y}) : \|\hat{y}\| \in [\varepsilon, 2r_\varepsilon]\}$ where

$$\begin{aligned} \tilde{F}^+_{\alpha,\tau}(\hat{y}) &= \varepsilon\left(1 - \eta(\|\hat{y}\|/r_\varepsilon)\right) F(\|\hat{y}\|/\varepsilon) + \eta(\|\hat{y}\|/r_\varepsilon)\mathcal{G}(\|\hat{y}\|) \\ \tilde{F}^-_{\alpha,\tau}(\hat{y}) &= -\varepsilon\left(1 - \eta(\|\hat{y}\|/r_\varepsilon)\right) F(\|\hat{y}\|/\varepsilon) - \eta(\|\hat{y}\|/r_\varepsilon)\mathcal{G}'(\|\hat{y}\|)\,. \end{aligned} \tag{9}$$

and $\eta : [0, \infty) \to \mathbb{R}$ is a smooth, monotone cut-off function satisfying $\eta(s) = 0$ for $s \in [0, 1/2]$ and $\eta(s) = 1$ for $s \in [2, \infty)$ while $r_\varepsilon := \varepsilon^{(3n-3)/(3n-2)}$.

All neighbouring perturbed hyperspheres corresponding to the hyperspheres in the initial configuration $\Lambda^\#[\alpha, \Gamma, \tau, \vec{\sigma}]$ can now be glued together by repeating the process outlined above.

Definition 2 The *approximate solution* with parameters $\tau, \vec{\sigma}$ is the hypersurface $\tilde{\Lambda}[\alpha,\Gamma,\tau,\vec{\sigma}]$ obtained from the gluing process above.

Note that by choosing the functions G_{sk} invariant under all $\rho \in G_\Gamma$ preserving $S^{sk}_\alpha[\vec{\sigma}_{sk}]$ and equal to $G_{s'k'}$ whenever there is $\rho \in G_\Gamma$ taking $S^{sk}_\alpha[\vec{\sigma}_{sk}]$ to $S^{s'k'}_\alpha[\vec{\sigma}_{s'k'}]$, then $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ becomes invariant under G_Γ as well.

3 The exact solution up to finite-dimensional error

3.1 The analytic set-up

Deforming the approximate solution. Since $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ is a hypersurface, it is possible to parametrize deformations of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ in a very standard way via normal deformations. These can be constructed by choosing a function $f : \tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}] \to \mathbb{R}$ and then considering the deformation $\phi_f : \tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}] \to \mathbb{S}^{n+1}$ given by $\phi_f(x) := \exp_x(f(x)\cdot N(x))$ where $\exp_x$ is the exponential map at the point x and $N(x)$ is the outward unit normal vector field of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ at the point x. For any given function f, the hypersurface $\phi_f(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}])$ is a normal graph over $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$, provided f is sufficiently small in a C^1 sense. Finding an exactly CMC normal graph near $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ therefore consists of finding a function f satisfying the equation $H_{\phi_f(\tilde{\Lambda}[\alpha,\Gamma,\tau,\vec{\sigma}])} = H_\alpha$, where H_Λ denotes the mean curvature of a hypersurface Λ.

Definition 3 Let $\Phi_{\tau,\vec{\sigma}}$ be the operator $f \longmapsto H_{\phi_f(\tilde{\Lambda}[\alpha,\Gamma,\tau,\vec{\sigma}])} - H_\alpha$.

This is a quasi-linear, second-order partial differential operator for the function f are a solution of $\Phi_{\tau,\vec{\sigma}}(f) = 0$ gives the desired deformation of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$.

The strategy of the proof. Finding a solution of the equation $\Phi_{\tau,\vec{\sigma}}(f) = 0$ when τ and $\vec{\sigma}$ are sufficiently small will be accomplished by invoking the *Banach space inverse function theorem* exactly as in [2]. As a reminder to the reader, this fundamental result will be stated here in fairly general terms [1].

Theorem (IFT) *Let $\Phi : X \to Z$ be a smooth map of Banach spaces, set $\Phi(0) := E$ and define the linearized operator $\mathcal{L} := \mathrm{D}\Phi(f) = \frac{\mathrm{d}}{\mathrm{d}s}\Phi(f + su)\big|_{s=0}$. Suppose $\mathcal{L}$ is bounded and surjective, possessing a bounded right inverse $\mathcal{R} : Z \to X$ satisfying*

$$\|\mathcal{R}(z)\| \leq C\|z\| \tag{10}$$

for all $z \in Z$. Choose R so that if $y \in B_R(0) \subseteq X$, then

$$\|\mathcal{L}(x) - \mathrm{D}\Phi(y)(x)\| \leq \frac{1}{2C}\|x\| \tag{11}$$

for all $x \in X$, where $C > 0$ is a constant. Then if $z \in Z$ is such that

$$\|z - E\| \leq \frac{R}{2C}, \tag{12}$$

there exists a unique $x \in B_R(0)$ so that $\Phi(x) = z$. Moreover, $\|x\| \leq 2C\|z - E\|$.

As in [2], it is not true that $\mathcal{L}_{\tau,\vec{\sigma}}$ is surjective with a bounded right inverse because of the obstruction caused by the *Jacobi fields* that generate an eigenspace of $\mathcal{L}_{\tau,\vec{\sigma}}$ whose corresponding eigenvalues tend to zero as $\tau \to 0$. A significant point of departure between this paper and [2] is that because of the lesser degree of symmetry considered here, the space of Jacobi fields that needs to be taken into account is much larger. The technique of projecting $\mathcal{L}_{\tau,\vec{\sigma}}$ onto a subspace of functions which is transverse to the co-kernel associated to suitable approximations of the Jacobi fields and then constructing a bounded right inverse for the projected linear operator will once again be used to resolve this difficulty. In conjunction with the IFT, this technique provides the solution of the CMC deformation problem *up to a finite-dimensional error term* lying in the span of the approximate Jacobi fields.

3.2 Function spaces and norms

The CMC deformation problem will once again be solved in *weighted Hölder spaces* since these properly determine the dependence on the parameter τ of the various estimates needed for the application of the IFT. A number of definitions are needed.

Definition 4 Identify the following regions of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$.

- Let $\mathcal{N}^{sk}$ be the *neck region* between the kth and $(k+1)$st perturbed hypersphere along the geodesic γ_s. Note that $\mathcal{N}^{sk}$ carries a scale parameter ε_{sk} depending smoothly on τ_s and $\vec{\sigma}_{sk}$ and $\vec{\sigma}_{s,k+1}$. In the stereographic coordinates $\mathcal{N}^{sk}$ is the set of points $(y^1, \hat{y})$ corresponding to $\|\hat{y}\| \leq r_{\varepsilon_{sk}}$.
- Let $\mathcal{T}^{sk,\pm}$ be the *transition regions* associated to the neck $\mathcal{N}^{sk}$. In the stereographic coordinates used to define this neck, $\mathcal{T}^{sk,+}$ is the set of points $(y^1, \hat{y})$ corresponding to $r_{\varepsilon_{sk}} < \|\hat{y}\| \leq 2r_{\varepsilon_{sk}}$ and $y^1 > 0$ whereas $\mathcal{T}^{sk,-}$ is the set of points $(y^1, \hat{y})$ corresponding to $r_{\varepsilon_{sk}} < \|\hat{y}\| \leq 2r_{\varepsilon_{sk}}$ and $y^1 < 0$.

- Let $\mathcal{E}^{sk}$ be the *spherical region* corresponding to the kth neck along the geodesic γ_s. This is the set of points in $\tilde{S}^{sk}_\alpha[\vec{a}_{sk}, \vec{\sigma}_{sk}] \setminus \bigcup_{j=1}^K B_r(p_j)$ where $p_1, \ldots, p_K$ are the points of $S^{sk}_\alpha[\vec{\sigma}_{sk}]$ closest to its neighbouring hyperspheres and r is a small radius chosen to exclude exactly the neck and transition region connecting $\tilde{S}^{sk}_\alpha[\vec{a}_{sk}, \vec{\sigma}_{sk}]$ to its neighbour near p_j.

Let $\mathcal{P} := \{p^\flat_{sk} : k = 0, \ldots, N_s - 1 \text{ and } s = 1, \ldots, L\}$ be the set of all points of $\mathbb{S}^{n+1}$ upon which the necks of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ are centered. Let K_{sk} denote the stereographic projection used to define the neck $\mathcal{N}^{sk}$. Fix some r_0 independent of τ such that the balls of radii $2r_0$ centered on any two points of $\mathcal{P}$ do not intersect.

Definition 5 The *weight function* $\zeta_\tau : \tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}] \to \mathbb{R}$ is defined by

$$\zeta_\tau(x) = \begin{cases} \varepsilon_{sk}\cosh(s) & x = K^{-1}_{sk}(\varepsilon_{sk}\psi(s), \varepsilon_{sk}\phi(s)\Theta) \in \mathcal{N}^{sk} \\ \mathit{Interpolation} & x \in \mathcal{T}^{sk} \\ \sqrt{\varepsilon^2_{sk} + \mathrm{dist}(x, p^\flat_{sk})^2} & x \in \tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}] \cap \left[\bar{B}_{r_0}(p^\flat_{sk}) \setminus \mathcal{T}^{sk,-} \cap \mathcal{N}^{sk} \cup \mathcal{T}^{sk,+}\right] \\ \mathit{Interpolation} & x \in \tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}] \cap \left[\bar{B}_{2r_0}(p^\flat_{sk}) \setminus B_{r_0}(p^\flat_{sk})\right] \\ 2r_0 & x \in \tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}] \setminus \bigcup_{\mathcal{P}} B_{2r_0}(p^\flat_{sk}) . \end{cases}$$

The interpolation is such that ζ_τ is smooth and monotone in the region of interpolation, and invariant under the group G_Γ.

Definition 6 Let $\mathcal{U} \subseteq \tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ and $\delta \in \mathbb{R}$ and $\beta \in (0, 1)$. The $C^{l,\beta}_\delta$ norm of a function defined on $\mathcal{U}$ is given by

$$|f|_{C^{l,\beta}_\delta(\mathcal{U})} := |f|_{l,\beta,\mathcal{U}\cap\mathcal{E}} + \sup_{\mathcal{P}} \sup_{r\in[0,r_0]} \left\{ \left(\sup_{x\in\mathcal{U}\cap\mathcal{A}^{sk}_r} [\zeta_\tau(x)]^{-\delta} \right) |f|_{l,\beta,\delta,\mathcal{U}\cap\mathcal{A}^{sk}_r} \right\} \tag{13}$$

using the notation of [2]. The Banach space $C^{k,\beta}_\delta(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}])$ denotes the $C^{l,\beta}$ functions of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ measured with respect to the norm (13), while $C^{l,\beta}_{\delta,sym}(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}])$ denotes functions $f \in C^{k,\beta}_\delta(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}])$ satisfying $f \circ \rho = f$ for all $\rho \in G_\Gamma$.

It is easy to deduce that $\Phi_{\tau,\vec{\sigma}} : C^{2,\beta}_\delta(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]) \to C^{0,\beta}_{\delta-2}(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}])$ is a well-defined and smooth operator and that $\mathcal{L}_{\tau,\vec{\sigma}} : C^{2,\beta}_\delta(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]) \to C^{0,\beta}_{\delta-2}(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}])$ is bounded in the operator norm by a constant independent of τ. Furthermore $\Phi_{\tau,\vec{\sigma}}$ and $\mathcal{L}_{\tau,\vec{\sigma}}$ can by symmetrized to yield new operators (which will be given the same names) on the symmetrized $C^{k,\beta}_\delta$ spaces.

3.3 Jacobi fields

The obstructions preventing the solvability of the equation $\Phi_{\tau,\vec{\sigma}}(f) = 0$ on $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ can be explained geometrically as follows. One can imagine transformations of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ which rotate exactly one of its constituent hyperspheres or catenoidal necks by a rotation in $SO(n+2)$ while leaving all the other constituent hyperspheres and necks fixed. The associated approximate Jacobi field is of the form χq_V where χ is a cut-off function supported on one constituent of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ and q_V is one of the exact Jacobi fields for this constituent. It is known that the linear span of these functions approximates the small eigenspaces of $\mathcal{L}_{\tau,\vec{\sigma}}$

well [5, Appendix B]. An explicit representation of the exact Jacobi fields on hypersphere and the catenoid will now be given.

1. Jacobi fields of the hyperspheres.

The linearized mean curvature operator of S_α is easily computed to be

$$\mathcal{L}_a := \sin^{-2}(\alpha)\,(\Delta_{\mathbb{S}^n} + n)\ .$$

The non-trivial rotations of S_α are generated by the vector fields

$$V_k := x^k \frac{\partial}{\partial x^0} - x^0 \frac{\partial}{\partial x^k} \quad \text{for} \quad k = 1, \ldots, n+1\,.$$

Taking the inner products of the V_k with the normal vector of S_α yields the Jacobi fields. They are the coordinate functions x^k restricted to S_α.

2. Jacobi fields of the catenoidal necks.

The linearized mean curvature operator of the standard catenoid $\Sigma \subseteq \mathbb{R}^{n+1}$ with respect to the Euclidean background metric is

$$\mathcal{L}_\Sigma := \frac{1}{\phi^n}\frac{\partial}{\partial s}\left(\phi^{n-2}\frac{\partial}{\partial s}\right) + \frac{1}{\phi^2}\Delta_{\mathbb{S}^{n-1}} + \frac{n(n-1)}{\phi^{2n}}$$

in its standard parametrization given in [2, Sect. 2.1]. The isometries generating the relevant Jacobi fields of Σ are as follows. First, the ambient space $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ possesses n translations along the $\mathbb{R}^n$ factor and one translation in the $\mathbb{R}$ direction, which are generated by the vector fields

$$V_k^{\text{trans}} := \frac{\partial}{\partial y^k} \quad \text{for} \quad k = 1, \ldots, n+1.$$

Then there are n rotations of $\mathbb{R} \times \mathbb{R}^n$ that do not preserve the $\mathbb{R}$-direction, which are generated by the vector fields

$$V_{1k}^{\text{rot}} := y^1 \frac{\partial}{\partial y^k} - y^k \frac{\partial}{\partial y^1} \quad \text{for} \quad k = 2, \ldots, n+1.$$

Finally, the motion of dilation in $\mathbb{R}^{n+1}$, though not an isometry, does preserve the mean curvature zero condition. Dilation is generated by the vector field

$$V^{\text{dil}} := \sum_{k=1}^{n+1} y^k \frac{\partial}{\partial y^k}\,.$$

By taking the inner product with the normal vector of Σ, one obtains the following non-trivial functions:

$$\begin{aligned}
J_1(s) &:= \langle N_\Sigma, V_1^{\text{trans}}\rangle = \frac{\dot\phi(s)}{\phi(s)} \\
J_k(s,\Theta) &:= \langle N_\Sigma, V_k^{\text{trans}}\rangle = -\frac{\Theta^k}{\phi^{n-1}(s)} \qquad k = 2,\ldots,n+1 \\
J_{1k}(s,\Theta) &:= \langle N_\Sigma, V_{1k}^{\text{rot}}\rangle = \Theta^k\left(\frac{\psi(s)}{\phi^{n-1}(s)} + \dot\phi(s)\right) \qquad k = 2,\ldots,n+1 \\
J_0(s) &:= \langle N_\Sigma, V^{\text{dil}}\rangle = \frac{\psi(s)\dot\phi(s)}{\phi(s)} - \frac{1}{\phi^{n-2}(s)}\,.
\end{aligned} \tag{14}$$

Note that the functions J_k with $k \neq 0$ have odd symmetry with respect to the central sphere of Σ, i.e. with respect to the transformation $s \mapsto -s$; while J_{1k} and J_0 have even symmetry.

Also J_1 is bounded while J_0 has linear growth in dimension $n = 2$ and is bounded in higher dimensions; J_k decays like $\exp(-(n-1)|s|)$ for large $|s|$; and J_{1k} grows like $\exp(|s|)$ for large $|s|$.

3.4 The linear analysis

The purpose of this section of the paper is to explicitly define the projected linearized mean curvature operator of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ and to find its right inverse on the appropriate Banach subspace of $C^{0,\beta}_{\delta-2,sym}(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}])$. The material that follows is quite similar to [2, Sect. 2.7] and important differences will be pointed out where appropriate.

The arguments that follow will require two carefully defined partitions of unity for the constituents of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$. First, for $s \in \{1, \ldots, L\}$ and $k \in \{0, \ldots, N_s - 1\}$, define the smooth cut-off functions

$$\eta^{sk}_{\text{neck}}(x) := \begin{cases} 1 & x \in \mathcal{N}^{sk} \\ \text{Interpolation} & x \in \mathcal{T}^{sk} \\ 0 & \text{elsewhere} \end{cases}$$

and for $s \in \{1, \ldots, L\}$ and $k \in \{0, \ldots, N_s\}$, define the smooth cut-off functions

$$\eta^{sk}_{\text{ext}}(x) := \begin{cases} 1 & x \in \mathcal{E}^{sk}_{\varepsilon} \\ \text{Interpolation} & x \in \text{ any adjoining } \mathcal{T}^{s'k'} \\ 0 & \text{elsewhere} \end{cases}$$

in such a way that $\sum_{s,k} \eta^{sk}_{\text{ext}} + \sum_{s,k} \eta^{sk}_{\text{neck}} = 1$. In addition, one can assume that these cut-off functions are invariant under the group of symmetries G_Γ and monotone in the interpolation regions. Second, set $r_\tau := \max_{s,k}\{r_{\varepsilon_{sk}}\}$ and for $s \in \{1, \ldots, L\}$ and $k \in \{0, \ldots, N_s - 1\}$ introduce the subsets $\mathcal{N}^{sk}(r) := \tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}] \cap B_r(p^\flat_{sk})$ where $r \in [r_\tau, r_0]$. This is a slightly enlarged version of the neck $\mathcal{N}^{sk}$ and its transition regions. Define the smooth cut-off functions

$$\chi^{sk}_{\text{neck},r}(x) := \begin{cases} 1 & x \in \mathcal{N}^{sk}(r) \\ \text{Interpolation} & x \in \mathcal{N}^{sk}(2r) \backslash \mathcal{N}^{sk}(r) \\ 0 & \text{elsewhere} \end{cases}$$

and for $s \in \{1, \ldots, L\}$ and $k \in \{0, \ldots, N_s\}$, define the smooth cut-off functions

$$\chi^{sk}_{\text{ext},r}(x) := \begin{cases} 1 & x \in \mathcal{E}^{sk} \backslash \left[\bigcup_{\text{adjoining}} \mathcal{N}^{s'k'}(2r) \right] \\ \text{Interpolation} & x \in \text{ any adjoining } \mathcal{N}^{s'k'}(2r) \backslash \mathcal{N}^{s'k'}(r) \\ 0 & \text{elsewhere} \end{cases}$$

so that once again $\sum_{s,k} \chi^{sk}_{\text{ext},r} + \sum_{s,k} \chi^{sk}_{\text{neck},r} = 1$ and invariance with respect to G_Γ as well as the monotonicity in the interpolation regions hold.

The cut-off functions above and the considerations of Sect. 3.3 leads to the definition of the space of approximate Jacobi fields of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ needed to construct the right inverse. Fix $r \in [r_\tau, r_0]$ to be small but independent of τ. Let x^t be the t^{th} coordinate function for $t = 1, \ldots, n$. For each s, k recall that $R_{sk}[\vec{\sigma}_{sk}]$ is the $SO(n+2)$-rotation bringing $S^{sk}_\alpha[\vec{\sigma}_{sk}]$ into S_α.

Definition 7 Define the following objects.

- The approximate Jacobi fields of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ are the functions

$$\tilde{q}^t_{sk} := \chi^{s,k}_{\text{ext},r} \cdot \left(x^t \left|_{\tilde{S}^{sk}_\alpha[\vec{a}_{sk}, \vec{\sigma}_{sk}]} \circ (R_{sk}[\vec{\sigma}_{sk}])^{-1} \right) \right. .$$

  Set $\tilde{\mathcal{K}} := \text{span}_{\mathbb{R}} \left\{ \tilde{q}^t_{sk} \quad \text{all} \quad s, t, k \right\}$.

- The set of G_Γ-invariant approximate Jacobi fields of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ is

$$\tilde{\mathcal{K}}_{\text{sym}} := \text{span}_{\mathbb{R}} \left\{ \tilde{q} \in \tilde{\mathcal{K}} \ : \ \tilde{q} \circ \rho = \tilde{q} \ \ \forall \rho \in G_\Gamma \right\}$$

- Denote the L^2-orthogonal complement of $\tilde{\mathcal{K}}_{\text{sym}}$ in $C^{l,\beta}_{\delta,\text{sym}}(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}])$ by $\left[C^{l,\beta}_{\delta,*}(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]) \right]^\perp$ and denote by

$$\pi : C^{l,\beta}_{\delta,\text{sym}}(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]) \to \left[C^{l,\beta}_{\delta,\text{sym}}(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]) \right]^\perp$$

  the corresponding L^2-projection operator.

The preliminary notation is in place and the key result of this section of the paper can now be stated and proved.

Proposition 8 *Suppose that the dimension of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ is $n \geq 3$ and choose $\delta \in (2-n, 0)$. If τ and $\|\vec{\sigma}\|$ are sufficiently small, then the operator*

$$\mathcal{L}^\perp_{\tau,\vec{\sigma}} : C^{2,\beta}_{\delta,sym}(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]) \to \left[C^{0,\beta}_{\delta-2,sym}(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]) \right]^\perp$$

possesses a bounded right inverse $\mathcal{R}_{\tau,\vec{\sigma}}$ satisfying the estimate

$$|\mathcal{R}_{\tau,\vec{\sigma}}(f)|_{C^{2,\beta}_\delta} \leq C |f|_{C^{0,\beta}_{\delta-2}}$$

where C is a constant independent of τ and $\vec{\sigma}$. If the dimension of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ is $n = 2$ then one can choose $\delta \in (-1, 0)$ and find a right inverse satisfying the estimate

$$|\mathcal{R}_{\tau,\vec{\sigma}}(f)|_{C^{2,\beta}_\delta} \leq C \varepsilon^\delta |f|_{C^{0,\beta}_{\delta-2}}$$

where $\varepsilon := \max_{s,k}\{\varepsilon_{sk}\}$ is the maximum of all the scale parameters of the necks of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ and C is a constant independent of τ and $\vec{\sigma}$.

Proof The proof of this result follows broadly the same plan as the proof in [2, Propositions 12, 13]. The significant differences occur in the first two steps, namely the derivation of the local solutions on the neck regions and the spherical regions of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$, while the other steps remain essentially unchanged. Thus only the first two steps will be given here in full detail, and moreover only in the dimension $n \geq 3$ case since the modifications needed for the $n = 2$ case can be readily adapted from [2, Proposition 13]. Furthermore, only the case $G_\Gamma = \{\text{Id}\}$ will be presented since the more general case simply amounts to additional book-keeping.

Suppose that $f \in \left[C^{0,\beta}_{\delta-2,sym}(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]) \right]^\perp$ is given. The solution of the equation $\mathcal{L}_{\tau,\vec{\sigma}}(u) = f$ will be constructed in three stages: local solutions on the neck regions will be found; then local solutions on the exterior regions will be found; and finally these solutions will be patched together to form an approximate solution which can be perturbed to a solution by iteration. To begin this process, write $f = \sum_{s,k} f^{sk}_{\text{ext}} + \sum_{s,k} f^{sk}_{\text{neck}}$ where $f^{sk}_{\text{ext}} := f \cdot \chi^{sk}_{\text{ext},r}$ and $f^{sk}_{\text{neck}} := f \cdot \chi^{sk}_{\text{neck},r}$.

Step 1. Local solutions on the neck regions. Consider a given neck $\mathcal{N} := \mathcal{N}^{sk}$ and for the moment, drop the super- and sub-scripted sk notation for convenience. The function $f_{\text{neck}} := f^{sk}_{\text{neck}}$ and the equation $\mathcal{L}_{\tau,\vec{\sigma}}(u) = f_{\text{neck}}$ can be pulled back to the scaled catenoid $\varepsilon\Sigma$ which carries a perturbation of the catenoid metric $4\varepsilon^2 g_\Sigma$. In this formulation, one can view f_{neck} as a function of compact support on $\varepsilon\Sigma$. The equation that will be solved in this step is $\frac{1}{4}\mathcal{L}_{\varepsilon\Sigma}(u) = f_{\text{neck}}$ where $\frac{1}{4}\mathcal{L}_{\varepsilon\Sigma}$ is the linearized mean curvature operator of $\varepsilon\Sigma$ carrying exactly the metric $4\varepsilon^2 g_\Sigma$.

By the theory of the Laplace operator on asymptotically cylindrical manifolds, there is a solution $u_{\text{neck}} \in C^{2,\beta}_\delta(\varepsilon\Sigma)$ that satisfies $\frac{1}{4}\mathcal{L}_{\varepsilon\Sigma}(u_{\text{neck}}) = (f_{\text{neck}})^\sharp$, where $(\cdot)^\sharp$ denotes the L^2-orthogonal projection onto the L^2-orthogonal complement of the linear span of the approximate Jacobi fields and the norm is the standard weighted norm on $\varepsilon\Sigma$. One can write

$$(f_{\text{neck}})^\sharp = f_{\text{neck}} + \sum_{t=1}^{n}(\lambda_t^+ \tilde{Q}^{t,+} + \lambda_t^- \tilde{Q}^{t,-})$$

where $\tilde{Q}^{t,+}$ and $\tilde{Q}^{t,-}$ are the pull-backs of the functions $\chi^{s,k+1}_{\text{neck},2r}\tilde{q}^t_{s,k+1}$ and $\chi^{sk}_{\text{neck},2r}\tilde{q}^t_{sk}$ to $\varepsilon\Sigma$, and

$$\lambda_t^\pm := -\frac{\int_{\varepsilon\Sigma} f_{\text{neck}} \cdot J_t}{\int_{\varepsilon\Sigma} \tilde{Q}^{t,\pm} \cdot J_t}.$$

One can check that $|\lambda_t^\pm| \le C\varepsilon^{\delta-2+n}|f|_{C^{0,\beta}_{\delta-2}(\varepsilon\Sigma)} \le C\varepsilon^{\delta-2+n}|f|_{C^{0,\beta}_{\delta-2}}$ where C is a constant independent of ε. Hence the estimate $|u_{\text{neck}}|_{C^{2,\beta}_\delta(\varepsilon\Sigma)} \le C|(f_{\text{neck}})^\sharp|_{C^{0,\beta}_{\delta-2}(\varepsilon\Sigma)} \le C|f|_{C^{0,\beta}_{\delta-2}}$ is valid, where C is also independent of ε. Finally, the function u_{neck} can be extended to all of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ by defining $\bar{u}^{sk}_{\text{neck}} := \chi^{sk}_{\text{neck},r} \cdot u_{\text{neck}}$. One has the estimate $|\bar{u}^{sk}_{\text{neck}}|_{C^{2,\beta}_\delta} \le C|f|_{C^{0,\beta}_{\delta-2}}$.

Step 2. Local solutions on the exterior regions. Consider a given spherical region $\mathcal{E} := \mathcal{E}^{sk}$ and again, drop the super- and sub-scripted sk notation for convenience. Given the local solution $\bar{u}_{\text{neck}}$ constructed in the previous step, choose a small $\kappa \in (0, 1)$ and define $\hat{f}_{\text{ext}} := \hat{f}^{sk}_{\text{ext}}$ where

$$\hat{f}^{sk}_{\text{ext}} := \chi^{sk}_{\text{ext},\kappa r}\left(f - \mathcal{L}_{\tau,\vec{\sigma}}\left(\sum_{s',k'}\bar{u}^{s'k'}_{neck}\right)\right).$$

This function vanishes within an $\varepsilon := \varepsilon_{sk}$-independent distance from the union of all the neck regions associated to $\mathcal{E}$. Therefore one can determine without difficulty $|\hat{f}_{\text{ext}}|_{C^{0,\beta}} \le C_\kappa|f|_{C^{0,\beta}_{\delta-2}}$ for some constant C_κ that depends on κ and δ. Here, $|\cdot|_{C^{0,\beta}}$ is the un-weighted Schauder norm.

The function $\hat{f}_{\text{ext}}$ can be viewed as a function of compact support on the standard hypersphere S_α vanishing in the neighbourhood of certain points $\{p_1, \ldots, p_K\} \subseteq S_\alpha$. The metric carried by S_α in this identification is a perturbation of the standard induced metric $\sin^2(\alpha)g_{\mathbb{S}^n}$. The equation that will be solved here is $\mathcal{L}_\alpha(u) = \hat{f}_{\text{ext}}$ up to projection onto the approximate co-kernel, where $\mathcal{L}_\alpha$ is the linearized mean curvature operator of S_α when it carries the un-perturbed metric $\sin^2(\alpha)g_{\mathbb{S}^n}$.

Compute the quantities $\mu_{tt'} := \int_{S_\alpha}\tilde{q}_t \cdot x^{t'}\big|_{S_\alpha}$ where $\tilde{q}_t \in \tilde{\mathcal{K}}$ are the Jacobi fields supported on $\tilde{S}_\alpha[\vec{a}, \vec{\sigma}]$ and pulled back to S_α and $x^t\big|_{S_\alpha}$ are the coordinate functions restricted to S_α.

Set $\mu^{tt'}$ equal to the components of the inverse of the matrix whose components are $\mu_{tt'}$. (It can easily be verified that this matrix is invertible because $\tilde{q}_t$ is almost equal to $x^t\,|_{S_\alpha}$ and these functions form an L^2-orthogonal set.) Now

$$\left(\hat{f}_{\text{ext}}\right)^\sharp := \hat{f}_{\text{ext}} - \sum_{t,t'} \tilde{q}_{t'} \cdot \mu^{tt'} \cdot \int_{S_\alpha} \hat{f}_{\text{ext}} \cdot x^t\,|_{S_\alpha}$$

is orthogonal to the coordinate functions restricted to S_α. The equation $\mathcal{L}_\alpha(u_{\text{ext}}) = \left(\hat{f}_{\text{ext}}\right)^\sharp$ can now be solved for u_{ext} in $C^{2,\beta}(S_\alpha)$. The solution satisfies the estimate $|u_{\text{ext}}|_{C^{2,\beta}(S_\alpha)} \leq C_\kappa |(\hat{f}_{\text{ext}})^\sharp|_{C^{0,\beta}_{\delta-2}(S_\alpha)}$.

The function u_{ext} can be extended to all of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ as follows. Suppose that $u_{\text{ext}}(p_j) := a_j$ for $j = 1, \ldots K$ and let $A_{\text{ext}} : S_\alpha \to \mathbb{R}$ be a smooth function that is locally constant near each p_j satisfying $A_{\text{ext}}(p_j) = a_j$. Then $u_{\text{ext}} = A_{\text{ext}} + \tilde{u}_{\text{ext}}$ where $\tilde{u}_{\text{ext}}$ is smooth function satisfying $\tilde{u}_{\text{ext}} = \mathcal{O}(\text{dist}(\cdot, p_j))$ near each p_j. For $j = 1, \ldots, K$, let $\mathcal{J}_j$ be the linear combination of the Jacobi fields J_0 and J_1 defined on the neck adjoining S_α at the point p_j that has limit a_j on the end of this neck attached to S_α and has limit zero on the other end of this neck. Note that $\mathcal{J}_j = a_j + \tilde{\mathcal{J}}_j$ where $\tilde{\mathcal{J}}_j = \mathcal{O}(\text{dist}(\cdot, p_j))$ in the part of this neck overlapping with S_α. Now define

$$\bar{u}^{sk}_{\text{ext}} := \eta_{\text{ext}} u_{\text{ext}} + \sum_{j=1}^{K} \eta^j_{\text{neck}} \mathcal{J}_j.$$

The extended function $\bar{u}_{\text{ext}}$ satisfies the estimate $|\bar{u}_{\text{ext}}|_{C^{2,\beta}_\delta} \leq C_\kappa |(\hat{f}_{\text{ext}})^\sharp|_{C^{0,\beta}_{\delta-2}}$ for some constant C_κ depending on κ and δ but not ε.

Step 3. Estimates and convergence. Define the function $\bar{u} := \sum_{s,k} \bar{u}^{sk}_{\text{neck}} + \sum_{s,k} \bar{u}^{sk}_{\text{ext}}$. Then computations along the lines of those found in Step 3 of [2, Propositions 12, 13] shows that

$$|\mathcal{L}^\perp_{\tau,\vec{\sigma}}(\bar{u}) - f|_{C^{0,\beta}_{\delta-2}} \leq \frac{1}{2} |f|_{C^{0,\beta}_{\delta-2}} \quad \text{and} \quad |\bar{u}|_{C^{2,\beta}_\delta} \leq C |f|_{C^{0,\beta}_{\delta-2}}.$$

The proof of the proposition now follows by a standard iteration argument. □

3.5 The solution of the non-linear problem up to finite-dimensional error

In order to apply the IFT to the CMC deformation problem, it remains to show that $\pi \circ \Phi_{\tau,\vec{\sigma}}(0)$ has small $C^{0,\beta}_{\delta-2}$ norm; and it is necessary to show that $\mathrm{D}(\pi \circ \Phi_{\tau,\vec{\sigma}})(f) - \mathcal{L}^\perp_{\tau,\vec{\sigma}}$ can be made to have small $C^{2,\beta}_\delta$-operator norm if f is chosen to have sufficiently small $C^{2,\beta}_\delta$ norm. These two estimates are in most respects identical to those computed in [2] and will thus only be sketched here.

Proposition 9 *The quantity $\pi \circ \Phi_{\tau,\vec{\sigma}}(0)$ satisfies the following estimate. If τ and $\vec{\sigma}$ are sufficiently small, then there exists a constant C independent of τ and $\vec{\sigma}$ so that*

$$|\pi \circ \Phi_{\tau,\vec{\sigma}}(0)|_{C^{0,\beta}_{\delta-2}} \leq C r_\varepsilon^{2-\delta} \tag{15}$$

where $\varepsilon := \max\{\varepsilon_{sk}\}$ is the maximum of all the scale parameters of the necks of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ and $r_\varepsilon := \varepsilon^{(3n-3)/(3n-2)}$.

Proof The estimate (15) can be computed as in [2] by verifying separately in the spherical regions, in the transition regions, and in the neck regions of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ that the mean curvature is sufficiently close to H_α, except with one significant modification in the first of these computations. To see this, consider one fixed spherical region $\mathcal{E}^{sk}$ pulled back to the standard hypersphere S_α. The expression for the mean curvature of a normal graph over S_α when the graphing function is $G := G_{sk}$, as given in [2], reads

$$H\left(\exp(GN_\alpha)(S_\alpha)\right) - H_\alpha = \frac{-\Delta G + n\sin(\alpha+G)\cos(\alpha+G)}{A\sin(\alpha+G)} - \frac{\nabla^2 G(\nabla G, \nabla G) - \cos(\alpha+G)\sin(\alpha+G)\|\nabla G\|^2}{A^3\sin(\alpha+G)} - H_\alpha \tag{16}$$

where ∇ and Δ are the covariant derivative and the Laplacian of the standard metric of $\mathbb{S}^n$, and $A = \left(\sin^2(\alpha+G) + \|\nabla G\|^2\right)^{1/2}$. By formally expanding this expression in when G is small as in [2], one finds that the largest term is $-(\Delta+n)(G)$. The quantity $(\Delta+n)(G)$ equals a term in $\tilde{\mathcal{K}}_{\mathrm{sym}}$ by definition, and it disappears under L^2 projection. The desired estimate follows as in [2]. □

Proposition 10 *The linearized mean curvature operator satisfies the following general estimate. If τ and $\vec{\sigma}$ are sufficiently small and $f \in C^{2,\beta}_{\delta,sym}(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}])$ has sufficiently small $C^{2,\beta}_\delta$ norm, then there exists a constant C independent of τ and $\vec{\sigma}$ so that*

$$\left| \mathrm{D}(\pi\circ\Phi_{\alpha,\tau\vec{\sigma}})(f)(u) - \mathcal{L}^{\perp}_{\tau,\vec{\sigma}}(u) \right|_{C^{0,\beta}_{\delta-2}} \leq C\varepsilon^{\delta-1}|f|_{C^{2,\beta}_\delta}|u|_{C^{2,\beta}_\delta} \tag{17}$$

for any function $u \in C^{2,\beta}_{\delta,sym}(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}])$, where $\varepsilon := \max\{\varepsilon_{sk}\}$ is the maximum of all the scale parameters of the necks of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$.

Proof This follows from a scaling argument exactly as in [2]. □

The non-linear estimates derived above, coupled with the construction of the right inverse and its estimate carried out in the previous sections now yield a solution of the equation $\pi\circ\Phi_{\tau,\vec{\sigma}}(f) = 0$ up to a finite-dimensional error term contained in the kernel of π.

Proposition 11 *If τ and $\vec{\sigma}$ are sufficiently small, then there exists $f_{\tau,\vec{\sigma}} \in C^{2,\beta}_\delta(X)$ satisfying $\pi\circ\Phi_{\tau,\vec{\sigma}}(f_{\alpha,\tau,\vec{\sigma}}) = 0$ and there exists a constant C independent of τ and $\vec{\sigma}$ so that*

$$|f_{\tau,\vec{\sigma}}|_{C^{2,\beta}_\delta} \leq C(\varepsilon)\cdot r_\varepsilon^{2-\delta}$$

where $C(\varepsilon) = \mathcal{O}(1)$ in dimension $n \geq 3$ and $C(\varepsilon) = \mathcal{O}(\varepsilon^\delta)$ in dimension $n = 2$. Here $\varepsilon := \max\{\varepsilon_{sk}\}$ is the maximum of all the scale parameters of the necks of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ and $r_\varepsilon := \varepsilon^{(3n-3)/(3n-2)}$. As a result, the hypersurface obtained by deforming $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ in the normal direction by an amount determined by $f_{\tau,\vec{\sigma}}$ is embedded if $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ is embedded.

Proof This follows exactly as in [2]. □

4 Solution of the finite-dimensional problem

4.1 The balancing map

Proposition 11 shows that the equation $\Phi_{\tau,\vec{\sigma}}(f) = 0$ can be solved up to a finite dimensional error term; i.e. a function $f_{\tau,\vec{\sigma}} \in C^{2,\beta}_{\delta,sym}(\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}])$ can be found so that only the L^2-projection of $\Phi_{\tau,\vec{\sigma}}(f_{\tau,\vec{\sigma}})$ to the subspace $\tilde{\mathcal{K}}_{\text{sym}}$ fails to vanish identically. Since there is such a function for each sufficiently small $\vec{\sigma} \in \mathcal{D}_\Gamma$, one can consider the map $\vec{\sigma} \mapsto \Phi_{\tau,\vec{\sigma}}(f_{\tau,\vec{\sigma}})$ as a function of $\vec{\sigma}$. It will now be shown that under the hypotheses of Main Theorem 1 there is a special choice of $\vec{\sigma}$ for which $\Phi_{\tau,\vec{\sigma}}(f_{\tau,\vec{\sigma}})$ vanishes completely. Therefore the solution $f_{\tau,\vec{\sigma}}$ for this choice of $\vec{\sigma}$ yields the desired deformation of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ into an exactly CMC hypersurface. In order to show how this special value of $\vec{\sigma}$ is found, one must first understand in greater detail the relationship between $\vec{\sigma}$ and the quantity $(\text{id} - \pi) \circ \Phi_{\tau,\vec{\sigma}}(f_{\tau,\vec{\sigma}})$ where π is the L^2-projection onto $\tilde{\mathcal{K}}^{\perp}_{\text{sym}}$.

To analyze this relationship properly, the first step to re-phrase the problem slightly. Let $\tilde{q}_1, \ldots, \tilde{q}_N$ be a basis for $\tilde{\mathcal{K}}_{\text{sym}}$ constructed from an L^2-orthonormal basis for the eigenfunctions of $\Delta_{\mathbb{S}} + n$ on S_α as in Definition 7. Next, define a slightly different set of functions $\tilde{q}'_1, \ldots, \tilde{q}'_N$ obtained from the $\tilde{q}_1, \ldots, \tilde{q}_N$ by replacing each $\chi_{\text{ext},r}$ appearing in the definition of a $\tilde{q}_j$ with $\chi_{\text{ext},r_\varepsilon}$. As usual, here $\varepsilon := \max\{\varepsilon_{sk}\}$ and $r_\varepsilon := \varepsilon^{(3n-3)/(3n-2)}$ where ε_{sk} is the scale parameter of the k^{th} neck along the geodesic γ_s. Now one can decompose

$$(\text{id} - \pi) \circ \Phi_{\tau,\vec{\sigma}}(f_{\tau,\vec{\sigma}}) = \sum_{i,j=1}^{N} M^{ij}(\vec{\sigma}) \cdot B_i(\vec{\sigma}) \cdot \tilde{q}_j$$

where $M^{ij}(\vec{\sigma})$ are the coefficients of the inverse of the matrix with coefficients $\int_{\phi_{f_{\tau,\vec{\sigma}}}(\tilde{\Lambda}[\alpha,\Gamma,\tau,\vec{\sigma}])} \tilde{q}_i \cdot \tilde{q}'_j$ and $B_i : \mathcal{D}_\Gamma \to \mathbb{R}$ are real-valued functions of the displacement parameters defined by

$$B_j(\vec{\sigma}) := \int\limits_{\phi_{f_{\tau,\vec{\sigma}}}(\tilde{\Lambda}[\alpha,\Gamma,\tau,\vec{\sigma}])} \Phi_{\tau,\vec{\sigma}}(f_{\tau,\vec{\sigma}}) \cdot \tilde{q}'_j \tag{18}$$

where $\phi_{f_{\tau,\vec{\sigma}}}$ is the normal deformation corresponding to $f_{\tau,\vec{\sigma}}$. One can check that the matrix M^{ij} is a small perturbation of the identity matrix and is indeed invertible.

Definition 12 The *balancing map* of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ with respect to the chosen basis $\{\tilde{q}_1, \ldots \tilde{q}_N\}$ of $\tilde{\mathcal{K}}_{sym}$ is the function $B_\tau : \mathcal{D}_\Gamma \to \mathbb{R}^N$ given by

$$B_\tau(\vec{\sigma}) := (B_1(\vec{\sigma}), \ldots, B_N(\vec{\sigma})),$$

and each $B_j : \mathcal{D}_\Gamma \to \mathbb{R}$ is defined as in (18).

In terms of the balancing map, what remains to be done in order to prove Main Theorem 1 is to find a value of $\vec{\sigma}$ for which $B_\tau(\vec{\sigma}) = 0$.

4.2 Approximating the balancing map and its derivative

The approximate balancing map. The balancing map can be better understood by deriving an approximation of the map which is independent of $f_{\tau,\vec{\sigma}}$. To see how this is done, note that each $\tilde{q}'_j$ is a G_Γ-invariant linear combination of the approximate Jacobi fields in Definition

7, each of which is supported on exactly one of the constituent perturbed hyperspheres of $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$. Thus it suffices to find a good approximation of the function

$$B : \vec{\sigma} \mapsto \int_{\phi_{f_{\tau,\vec{\sigma}}}(\tilde{\Lambda}[\alpha,\Gamma,\tau,\vec{\sigma}])} \Phi_{\tau,\vec{\sigma}}(f_{\tau,\vec{\sigma}}) \cdot \tilde{q}'$$

where $\tilde{q}' = \sum_{t=1}^{n} a_t \chi^{sk}_{ext,r_\varepsilon} q^t_{sk}$ and q^t_{sk} are the Jacobi fields of this hypersphere as in Definition 7.

Suppose that the (s,k)-perturbed hypersphere in $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$ is a perturbation of $S^{sk}_\alpha[\vec{\sigma}_{sk}] := (R_{sk}[\vec{\sigma}_{sk}])^{-1}(S_\alpha \setminus \{p_1, \ldots, p_K\})$. Recall that the infinitesimal generator of rotation associated to q^t_{sk} is the vector field

$$Y^t_{sk} := (R_{sk}[\vec{\sigma}_{sk}])^{-1}_* \left[Y^t \circ (R_{sk}[\vec{\sigma}_{sk}]) \right] \qquad \text{where} \qquad Y^t := x^t \frac{\partial}{\partial x^0} - x^0 \frac{\partial}{\partial x^t} . \tag{19}$$

Set $Y := \sum_{t=1}^{n} a_t Y^t_{sk}$ and $q := \sum_{t=1}^{n} a_t q^t$. An analysis of the function B reveals the following.

Proposition 13 *Let $\tilde{q}$ be as above. Then the function B can be decomposed as*

$$B(\vec{\sigma}) = \mathring{B}(\vec{\sigma}) + E(\vec{\sigma}) .$$

In this decomposition, $\mathring{B} : \mathcal{D}_\Gamma \to \mathbb{R}$ is defined as follows. Suppose that $p_j := \exp_{p_0}(\alpha T_j)$ where T_j is the unit vector in $T_{p_0}\mathbb{S}^{n+1}$ tangent to the geodesic connecting p_0 and p_j. Then

$$\mathring{B}(\vec{\sigma}) := \sum_{j=1}^{K} \omega \varepsilon_j^{n-1} \langle T_j, Y \rangle . \tag{20}$$

Furthermore, $E : \mathcal{D}_\Gamma \to \mathbb{R}$ satisfies the estimate

$$\| E(\vec{\sigma}) \|_{C^2} \leq C r_\varepsilon^n$$

where C is a constant independent of τ and $\vec{\sigma}$.

Proof The integral defining B is invariant under rotation, so that one can assume that $R_{sk}[\vec{\sigma}_{sk}]$ is the identity so that B integrates over the standard punctured hypersphere $S_\alpha \setminus \{p_1, \ldots, p_K\}$, which shall be denoted here by $\tilde{S}^0[\vec{\sigma}_0]$. Denote the nearest neighbours of $\tilde{S}^0[\vec{\sigma}_0]$ by $\tilde{S}^j[\vec{\sigma}_j]$. Let these be connected to $\tilde{S}^0[\vec{\sigma}_0]$ through necks $\mathcal{N}_j$ with scale parameters ε_j. Finally, denote by D_j the disk $\{(0, \hat{y}) \in \mathbb{R} \times \mathbb{R}^n : \|\hat{y}\| \leq \varepsilon_j\}$ pushed forward by the canonical coordinate chart corresponding to the neck $\mathcal{N}_j$ and let $c_j = \partial D_j$. In other words, c_j is the smallest sphere in the throat of $\mathcal{N}_j$ and D_j is an n-dimensional cap for c_j. Denote by $\mathcal{N}_j^-$ the component of $\mathcal{N}_j \setminus c_j$ that is attached to $\tilde{S}^0[\vec{\sigma}_0]$ at the point p_j and set $\mathcal{N}^- := \mathcal{N}_1^- \cup \cdots \cup \mathcal{N}_K^-$.

Consider now the integral defining B. The idea is to apply the Korevaar–Kusner–Solomon and Kapouleas balancing formula (1) for the integral of $\Phi_{\tau,\vec{\sigma}}(f_{\tau,\vec{\sigma}}) := H_{\phi_{f_{\tau,\vec{\sigma}}}(\tilde{\Lambda}[\alpha,\Gamma,\tau,\vec{\sigma}])} - H_\alpha$ to replace this integral with a sum of boundary terms. Then the fact that $f_{\tau,\vec{\sigma}}$ is small gives an approximate expression that pertains solely to the initial configuration of hyperspheres. These calculations are

$$
\begin{aligned}
B(\vec{\sigma}) &= \int\limits_{\phi_{f_{\tau,\vec{\sigma}}}(\tilde{S}^0[\vec{\sigma}_0]\cup\mathcal{N}^-)} \Phi_{\tau,\vec{\sigma}}(f_{\tau,\vec{\sigma}})\cdot\chi_{ext,r_\varepsilon}\cdot q \\
&= \int\limits_{\phi_{f_{\tau,\vec{\sigma}}}(\tilde{S}^0[\vec{\sigma}_0]\cup\mathcal{N}^-)} \Phi_{\tau,\vec{\sigma}}(f_{\tau,\vec{\sigma}})\cdot q + \mathcal{O}(r_\varepsilon^n) \\
&= \sum_{j=1}^{K} \int\limits_{\phi_{f_{\tau,\vec{\sigma}}}(c_j)} \big\langle \nu_j, Y\big\rangle + \mathcal{O}(r_\varepsilon^n) \\
&= \sum_{j=1}^{K} \int\limits_{c_j} \big\langle \nu_j, Y\big\rangle + \mathcal{O}(\varepsilon^{\delta+n-1}|f|_{C^{2,\beta}_\delta}) + \mathcal{O}(r_\varepsilon^n)
\end{aligned} \tag{21}
$$

where ν_j is the outward unit normal vector field of c_j tangent to $\mathcal{N}^{j,-}$. Note that the $\int_{D_j}$ terms have been absorbed into the error term. This is because when ε is small then these quantities are much smaller than the $\int_{c_j}$ terms.

Finally, the calculation of the integrals $\int_{c_j}\langle\nu_j, Y\rangle$ in (21) can be carried out in the stereographic coordinate chart used to define $\mathcal{N}_j$. This is very straightforward and yields a quantity proportional to the $(n-1)$ dimensional area of c_j in the form $\omega\varepsilon_j^{n-1}\langle\dot{\gamma}_j(\alpha), Y\rangle$ where γ_j is the geodesic from p_0 to p_j while ω is a constant independent of ε. But since Y is a Killing field, this quantity remains constant along γ_j and can thus be transported to p_0. The desired formulæ follow. □

The calculations of the previous proposition show that the balancing map consists of a collection of principal terms like (20), one for each perturbed hypersphere in $\tilde{\Lambda}[\alpha, \Gamma, \tau, \vec{\sigma}]$, plus error terms which are of size $\mathcal{O}(r_\varepsilon^n)$. The principal term corresponding to a given perturbed hypersphere depends on the displacement parameter of this perturbed hypersphere, as well as on the displacement parameters of all neighbouring perturbed hyperspheres. It is important to realize that the principal term depends on no other displacement parameters. As defined in the introduction, an initial configuration of hyperspheres is *approximately balanced* if $\mathring{B}(0) = 0$.

The derivative of the approximate balancing map. A formula for the derivative of the approximate balancing map at $\vec{\sigma} = 0$ will also be needed in the sequel. To this end, a more explicit formula illustrating the dependence of $\mathring{B}$ on $\vec{\sigma}$ is needed. In what follows, denote once again the (s,k)-perturbed hypersphere by $\tilde{S}^0[\vec{\sigma}_0]$ and suppose it is centered on $p_0[\vec{\sigma}_0]$. As before, one can assume that $\tilde{S}^0[0]$ is a perturbation of the punctured hypersphere $S_\alpha\backslash\{p_1, \ldots, p_K\}$. Denote the nearest neighbours of $\tilde{S}^0[\vec{\sigma}_0]$ by $\tilde{S}^j[\vec{\sigma}_j]$ for $j = 1, \ldots, K$ and suppose these are centered at $p_j[\vec{\sigma}_j]$ with $p_j[0] = p_j$. Denote the geodesic connecting $p_0[\vec{\sigma}_0]$ to $p_j[\vec{\sigma}_j]$ by $\gamma_j[\vec{\sigma}_0, \vec{\sigma}_j]$. Let the tangent vectors of $\gamma_j[\vec{\sigma}_0, \vec{\sigma}_j]$ at $p_0[\vec{\sigma}_0]$ and $p_j[\vec{\sigma}_j]$ be $T_j[\vec{\sigma}_0, \vec{\sigma}_j] := \csc(\tau_j + 2\alpha)\big(p_j[\vec{\sigma}_j] - p_0[\vec{\sigma}_0]\cos(\tau_j + 2\alpha)\big)$ and $T_j'[\vec{\sigma}_0, \vec{\sigma}_j] := \csc(\tau_j + 2\alpha)\big(p_0[\vec{\sigma}_0] - p_j[\vec{\sigma}_j]\cos(\tau_j + 2\alpha)\big)$.

The map $\mathring{B}$ can be related to $\vec{\sigma}$ explicitly as follows. First, the relationship between the scale of the neck used to connect two perturbed hyperspheres and their separation (found in Sect. 2.4) gives $\varepsilon_j := \varepsilon(\tau_j)$ where $\tau_j := \mathrm{dist}(p_0[\vec{\sigma}_0], p_j[\vec{\sigma}_j]) - 2\alpha$ and $\varepsilon : \mathbb{R} \to \mathbb{R}$ is some universal function determined via the matching process. Recall further that $\tilde{S}^j[\vec{\sigma}_j] = \mathcal{W}_{\vec{\sigma}_j}\big(S^j[0]\big)$ for $j = 0, \ldots, K$ where $\mathcal{W}_{\vec{\sigma}_j}$ is the unique $SO(n+2)$-rotation that coincides with the exponential map at $p_j[0]$ in the direction of $\vec{\sigma}_j$. Moreover, the basis of infinitesimal generators of the rotations of $\tilde{S}^0[\vec{\sigma}_0]$ are of the form $\left(\mathcal{W}_{\vec{\sigma}_0}\right)_* Y \circ \mathcal{W}_{\vec{\sigma}_0}^{-1}$ where Y is a linear combinations of the vector fields given in (19). One therefore obtains the formula

$$\mathring{B}(\vec{\sigma}_0, \vec{\sigma}_1, \ldots, \vec{\sigma}_K) = \sum_{j=1}^{K} \omega \varepsilon_j^{n-1} \left\langle \left(\mathcal{W}_{\vec{\sigma}_0}\right)_*^{-1} T_j[\vec{\sigma}_0, \vec{\sigma}_j], Y \right\rangle \big|_{p_0[0]} \ . \tag{22}$$

This illustrates completely how $\mathring{B}$ depends only on $\vec{\sigma}_0$ and $\vec{\sigma}_j$ for $j = 1, \ldots, K$.

Proposition 14 *Let V be a tangent vector at the origin in the space of displacement parameters. Suppose that $V_0 \in T_{p_0[0]}\mathbb{S}^{n+1}$ is the component of V corresponding to the perturbed hypersphere $\tilde{S}^0[\vec{\sigma}_0]$ and $V_j \in T_{p_j[0]}\mathbb{S}^{n+1}$ are the components of V corresponding to the nearest neighbours $\tilde{S}^j[\vec{\sigma}_j]$ for $j = 1, \ldots, K$. Then*

$$\begin{aligned} \mathrm{D}\mathring{B}(0)(V) = &-\sum_{j=1}^{K} (n-1)\, \omega\, \varepsilon_j^{n-2} \dot{\varepsilon}(\tau_j) \left(\left\langle V_0^{\parallel_j}, Y \right\rangle - \tan(\tau_j + 2\alpha) \left\langle \left[V_j^{\sharp} \right]^{\parallel_j}, Y \right\rangle \right) \\ &-\sum_{j=0}^{K} \omega\, \varepsilon_j^{n-1} \left(\left\langle V_0^{\perp_j}, Y \right\rangle - \tan(\tau_j + 2\alpha) \left\langle \left[V_j^{\sharp} \right]^{\perp_j}, Y \right\rangle \right) \end{aligned} \tag{23}$$

where $X^{\parallel_j}$ and $X^{\perp_j}$ denote the projections of a vector X parallel and perpendicular to $T_j[0, 0]$ while

$$V_j^{\sharp} := \frac{V_j - p_0[0]\langle p_0[0], V_j \rangle}{\sin(\tau_j + 2\alpha)}$$

is the re-scaled orthogonal projection of V_j into $T_{p_0[0]}\mathbb{S}^{n+1}$.

Proof The various terms in the formula (22) for $\mathring{B}(\vec{\sigma})$ must be differentiated at $\vec{\sigma} = 0$. Let $\vec{\sigma}_0(t) = tV_0$ and $\vec{\sigma}_j(t) = tV_j$ be paths in the displacement parameter space, where V_0 and V_j are considered as vectors in $T_{p_0[0]}\mathbb{S}^{n+1}$ and $T_{p_j[0]}\mathbb{S}^{n+1}$ respectively. First,

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varepsilon_j(tV_0, tV_j) &= \dot{\varepsilon}(\tau_j) \cdot \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \arccos\left(\langle p_0[tV_0], p_j[tV_j] \rangle \right) \\ &= -\frac{\dot{\varepsilon}_j(\tau_j) \cdot \left(\langle V_0, p_j[0] \rangle + \langle p_0[0], V_j \rangle \right)}{\sqrt{1 - \langle p_0[0], p_j[0] \rangle^2}} \, . \end{aligned}$$

The first term in the formula for $\mathrm{D}\mathring{B}(0)(V)$ involving the parallel parts of V_0 and V_j follows from this using the formula for $T_j[0, 0]$ as well as $\langle p_0[0], V_0 \rangle = \langle p_j[0], V_j \rangle = 0$.

Next, realize that $\left(\mathcal{W}_{\vec{\sigma}_0}\right)_*^{-1} T_j[\vec{\sigma}_0, \vec{\sigma}_j]$ is the tangent vector of the geodesic connecting the point $p_0[0]$ to $\mathcal{W}_{\vec{\sigma}_0}^{-1} \circ \mathcal{W}_{\vec{\sigma}_j}(p_j[0])$ at $p_0[0]$. A calculation reveals

$$\left(\mathcal{W}_{\vec{\sigma}_0}\right)_*^{-1} T_j[\vec{\sigma}_0, \vec{\sigma}_j] = \frac{\mathcal{W}_{\vec{\sigma}_0}^{-1} \circ \mathcal{W}_{\vec{\sigma}_j}(p_j[0]) - \left\langle \mathcal{W}_{\vec{\sigma}_0}^{-1} \circ \mathcal{W}_{\vec{\sigma}_j}(p_j[0]),\ p_0[0] \right\rangle \cdot p_0[0]}{\sqrt{1 - \left\langle \mathcal{W}_{\vec{\sigma}_0}^{-1} \circ \mathcal{W}_{\vec{\sigma}_j}(p_j[0]),\ p_0[0] \right\rangle^2}}.$$

Together with the definition of $\mathcal{W}_{\vec{\sigma}_j}$ one then finds after some work

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\left(\mathcal{W}_{tV_0}\right)_*^{-1} T_j[tV_0, tV_j] = \frac{T_j[0,0]\cdot\big(\langle V_j, p_0[0]\rangle + \langle V_0, p_j[0]\rangle\big)\cdot\langle p_j[0], p_0[0]\rangle}{1-\langle p_j[0], p_0[0]\rangle^2} + V_j^\sharp - \frac{V_0\cdot\langle p_j[0], p_0[0]\rangle}{\sqrt{1-\langle p_j[0], p_0[0]\rangle^2}}.$$

The second term in the formula for $\mathrm{D}\mathring{B}(0)(V)$ involving the transverse parts of V_0, $V_j^\sharp$ follows. □

4.3 Conclusion of the proof of Main Theorem 1

The finite-dimensional inverse function theorem will be used to locate a zero of B_τ. The first step is to approximate B_τ by the simpler mapping $\mathring{B}_\tau : \mathcal{D}_\Gamma \to \mathbb{R}^K$ obtained by replacing each B_j term in (18) by the corresponding function $\mathring{B}_j : \mathcal{D}_\Gamma \to \mathbb{R}$ of the form (20). The mapping $\mathring{B}_\tau$ is independent of $f_{\tau,\vec{\sigma}}$ and therefore depends only on the geometry of initial configuration $\Lambda^\#[\alpha,\Gamma,\tau,\vec{\sigma}]$. The hypotheses of Main Theorem 1 assert that $\Lambda^\#[\alpha,\Gamma,\tau,\vec{\sigma}]$ is balanced, meaning that $\mathring{B}_\tau(0) = 0$. By Proposition 13, one now has $B_\tau(0) = E_\tau(0)$ where $E_\tau(\vec{\sigma}) := B_\tau(\vec{\sigma}) - \mathring{B}_\tau(\vec{\sigma})$. This error term satisfies $\|E_\tau(0)\| = \mathcal{O}(r_\varepsilon^n)$ which is smaller than the operator norm of $\mathrm{D}\mathring{B}(0)$. One can therefore attempt to use the finite-dimensional IFT to find a nearby $\vec{\sigma}$ so that $B_\tau(\vec{\sigma}) = 0$.

It is important to incorporate into the analysis the fact that B_τ can often not be a full-rank mapping. To see this why this is so, let $Y_1,\ldots,Y_d$ be a basis for the infinitesimal generators of one-parameter families of rotations of $\mathbb{S}^{n+1}$ that are equivariant with respect to the symmetries of $\tilde{\Lambda}[\alpha,\Gamma,\tau,\vec{\sigma}]$. This means $\rho_*(Y_j\circ\rho) = Y$ for all $\rho\in G_\Gamma$ and this ensures that the functions $\langle Y_j, \nu\rangle : \tilde{\Lambda}[\alpha,\Gamma,\tau,\vec{\sigma}] \to \mathbb{R}$, where ν is the outward unit normal of $\tilde{\Lambda}[\alpha,\Gamma,\tau,\vec{\sigma}]$, are invariant with respect to G_Γ. Now the first variation formula for the volume of hypersurfaces, applied to the volume-preserving deformation given by rotation in the Y_j direction, leads to the equation

$$\int_{\phi_f(\tilde{\Lambda}[\alpha,\Gamma,\tau,\vec{\sigma}])} \Phi_{\tau,\vec{\sigma}}(f)\cdot\langle\nu, Y_j\rangle = 0 \quad \forall\, j = 1,\ldots,d$$

where ν_f is the unit outward normal vector field of $\phi_f(\tilde{\Lambda}[\alpha,\Gamma,\tau,\vec{\sigma}])$. Therefore one sees that there are maps $\mathcal{Y}_j : \mathcal{D}_\Gamma \to \mathbb{R}^K$ for $j = 1,\ldots,d$ with

$$\big[B_\tau(\vec{\sigma})\big]\cdot\big[\mathcal{Y}_j(\vec{\sigma})\big] = 0 \quad \forall\, j = 1,\ldots,d \tag{24}$$

where $\cdot$ denotes the Euclidean inner product. Hence the rank of B_τ is at most $K-d$.

The correct interpretation of (24) is to say that the graph $\{(\vec{\sigma}, B_\tau(\vec{\sigma})) : \vec{\sigma}\in\mathcal{D}_\Gamma\}$ is contained in the submanifold $\{(\vec{\sigma}, b) : b\cdot\mathcal{Y}_1(\vec{\sigma}) = \cdots = b\cdot\mathcal{Y}_d(\vec{\sigma}) = 0\}$ of $\mathcal{D}_\Gamma\times\mathbb{R}^K$. Therefore it suffices to show that the equation $pr\circ B_\tau(\vec{\sigma}) = 0$ has a solution, where pr is the orthogonal projection to the orthogonal complement of the subspace spanned by $\mathcal{Y}_1(0),\ldots,\mathcal{Y}_d(0)$. Note that the linearization of $pr\circ B_\tau$ at zero maps *into* this orthogonal complement, and thus $\mathrm{D}\,(pr\circ B_\tau)\,(0) = \mathrm{D}B_\tau(0)$. In addition, the calculations of the proof of Proposition 13 show that $(id - pr)\circ\mathrm{D}\mathring{B}_\tau(0) = L$ where L is a linear operator with $\mathcal{O}(r_\varepsilon^n)$ coefficients.

The hypotheses of Main Theorem 1 assert that $\Lambda^{\#}[\alpha, \Gamma, \tau, \vec{\sigma}]$ has the property that $\mathrm{D}\mathring{B}_\tau(0)$ has full rank. Hence $(id - pr) \circ \mathrm{D}\mathring{B}_\tau(0)$ and $\mathrm{D}B_\tau(0)$ do as well. Furthermore, the operator norm of $\mathrm{D}\mathring{B}_\tau(0)$ is $\mathcal{O}(C(\varepsilon)\varepsilon^{n-1})$. Hence $B_\tau(\vec{\sigma}) = b$ will be solvable for b inside a ball centered on $pr \circ E_\tau(0)$ whose radius is $\mathcal{O}(C(\varepsilon)\varepsilon^{n-1})$. When ε is sufficiently small, 0 is contained within this ball. Hence the equation $B_\tau(\vec{\sigma}) = 0$ is solvable for small $\vec{\sigma} \in \mathcal{D}_\Gamma$. The proof of Main Theorem 1 is complete. □

5 Applications of the balancing formulæ

5.1 A simple example

A simple example serves both to develop intuition for the approximate balancing map (20) and its derivative (23), as well as to show that the kernel of the derivative of the approximate balancing map can be quite large in the absence of symmetries. While this feature is also present in the CMC gluing construction in Euclidean space, it is here much more restrictive because the trick of imposing decay conditions at infinity to reduce the size of the kernel of the Euclidean analogue of the approximate balancing map is not available. Therefore one must impose symmetry conditions or else expect to work quite hard to find an initial configuration of hyperspheres that can be glued together and perturbed into an exactly CMC hypersurface using the gluing technique.

Consider exactly one geodesic, without loss of generality the (x^0, x^1)-equator γ, and let R^{01}_θ be the rotation by an angle θ in the (x^0, x^1)-plane that translates along γ. Position N hyperspheres of radius $\cos(\alpha)$ around γ, separated by a distance of τ from each other, so that $(\tau + 2\alpha)N = 2\pi m$ for some integer m. These hyperspheres are of the form $S^k_\alpha := \left(R^{01}_{\tau+2\alpha)}\right)^k (S_\alpha)$ which are centered at $p_k := \gamma((\tau + 2\alpha)k)$. Let $\Lambda^{\#} := \bigcup_{k=0}^{N-1} S^k_\alpha$. Note that this initial configuration is balanced because the vanishing of the approximate balancing map is equivalent to the equal spacing of the hyperspheres along a single geodesic.

The initial configuration $\Lambda^{\#}$ yields the Delaunay-like hyperspheres in Butscher's paper [2] using the gluing technique together with imposing as many symmetries as possible on the deformations. Now, however, no symmetries will be imposed and as a result the approximate balancing map becomes non-trivial. In the absence of any symmetry conditions constraining the displacement parameters of $\Lambda^{\#}$, there are n displacement parameters for each hypersphere in $\Lambda^{\#}$. For each hypersphere S^k_α, these will be decomposed into one displacement parameter corresponding to the displacement of S^k_α along γ and $n-1$ displacement parameters corresponding to the displacement of S^k_α perpendicular to γ. To parametrize these displacement parameters in a uniform way, note that $T_{p_k}\mathbb{S}^{n+1}$ is spanned by $T_k := \dot{\gamma}((\tau + 2\alpha)k)$ and $\frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}$. Thus one can set

$$\vec{\sigma}^k := \sigma^k_1 T_k + \sum_{j=2}^{n} \sigma^k_j \frac{\partial}{\partial x^j}$$

as the displacement parameter for S^k_α. Note that $\vec{\sigma}^0 = \vec{\sigma}^N$ by periodicity.

It will now be shown that the kernel of the derivative of the approximate balancing map is very large. Note that Main Theorem 1 still applies because each element in the kernel of $\mathrm{D}\mathring{B}(0)$ is induced from a rotation of $\mathbb{S}^{n+1}$. Let $\vec{V}^k := (V^k_1, V^k_2, \ldots, V^k_n)$ denote infinitesimal displacements satisfying $\vec{V}^0 = \vec{V}^N$. To compute $\mathrm{D}\mathring{B}(0, \ldots, 0)(\vec{V}^1, \ldots, \vec{V}^N)$ one needs formulæ for the re-scaled orthonormal projection operators $X \mapsto X^\sharp$ that appear there. It is easy to deduce

$$
\begin{aligned}
(V_1^{k\pm 1})^{\sharp_k} &= \frac{V_1^k}{\tan(\tau + 2\alpha)}\, T_k \\
(V_j^{k\pm 1})^{\sharp_k} &= \frac{V_j^k}{\sin(\tau + 2\alpha)}\, \frac{\partial}{\partial x^j} \qquad j = 2, \ldots, n
\end{aligned}
$$

Consequently, the derivative of the approximate balancing map on the k^{th} perturbed hypersphere takes the form

$$
\begin{aligned}
\mathrm{D}\mathring{B}(0)(\vec{V}^{k-1}, \vec{V}^{k}, \vec{V}^{k+1}) = -\omega\varepsilon^{n-2}\dot{\varepsilon}\left(2V_1^k - V_1^{k-1} - V_1^{k+1}\right) \\
- \omega\varepsilon^{n-1}\left(2V_j^k - \sec(\tau + 2\alpha)(V_j^{k-1} + V_j^{k+1})\right).
\end{aligned}
$$

The recursion formulæ $2V_1^k - V_1^{k-1} - V_1^{k+1} = 0$ and $2V_j^k - \sec(\tau + 2\alpha)(V_j^{k-1} + V_j^{k+1}) = 0$ for elements in the kernel of $\mathrm{D}\mathring{B}(0)$, together with the periodic boundary conditions $V_j^0 = V_j^N$ for all $j = 1, \ldots, N$, are easy to solve and yield

$$
\begin{aligned}
V_1^k &= 1 \quad \text{for all} \quad k = 0, \ldots, N \\
V_j^k &= \sin\left((\tau + 2\alpha)(k_0 + k)\right) \quad \text{for} \quad j = 2, \ldots, n \quad \text{and} \quad k_0 \in \{0, \ldots, N-1\}.
\end{aligned}
$$

There are either $\frac{1}{2}N(n-1) + 1$ or $N(n-1) + 1$ linearly independent solutions of this type, depending on whether N is even or odd. These solutions correspond to the change of displacement parameter induced by the rotation of $\mathbb{S}^{n+1}$ parallel to γ and transverse to γ.

5.2 An unachievable configuration

The intuition gained from the preceding example can be used to explain why a reasonably simple configuration, possessing an analogue in Euclidean space, cannot be achieved using the methods developed in this paper. The configuration in question consists of positioning hyperspheres around two intersecting geodesics that make an arbitrary angle to each other at the point of intersection. This is a slightly less symmetric version of the configuration considered in [2] where a CMC hypersurface is created from hyperspheres positioned around two orthogonally intersecting geodesics. One should note that the existence of a CMC hypersurface of this kind has not been ruled out; it is just that the techniques developed herein are not sufficient for its construction.

The reason the methods of this paper fail for the less symmetric configuration can be described as follows. First, let R_θ^{0j} be the rotation of the (x^0, x^j)-plane. Let γ_j be the (x^0, x^j)-equator for $j = 1, 2$. Choose $\alpha, \tau \in (0, \pi)$ and integers m, N so that $(\tau + 2\alpha)N = 2\pi m$. Also, choose N of the form $N = 4N_0$. The initial configuration in question, which shall be denoted $\Lambda_\theta^\#$, consists of the hyperspheres $S_\alpha^{2,k,\pm} := R_{\pm\theta}^{01} \circ (R_{\tau+2\alpha}^{02})^k (S_\alpha)$ for $k = 0, \ldots, N-1$. When $\theta \neq \pi/2$, the maximal symmetries one can impose on the deformations of the approximate solution constructed from $\Lambda_\theta^\#$ are: all orthogonal transformations of the $x^3, \ldots, x^{n+1}$ coordinates; and the reflections sending x^j to $-x^j$ and keeping all other coordinates fixed, for $j = 0, 1, 2$. As a result, there are two sets of invariant approximate Jacobi fields. These are: the translation of $S_\alpha^{2,k,+}$ along the geodesic $R_\theta^{01}(\gamma_2)$ for $k \in \{1, \ldots, N_0 - 1\}$ and extended by symmetry to $S_\alpha^{2,k,-}$, $S_\alpha^{2,-k,\pm}$ and $S_\alpha^{2,2N_0 \pm k,\pm}$; and the rotation of $S_\alpha^{2,k,+}$ in the (x^0, x^1)-plane transverse to $R_\theta^{02}(\gamma)$ and similarly extended by symmetry. None of these invariant approximate Jacobi fields are induced by rotations of $\mathbb{S}^{n+1}$. Note that there are no Jacobi fields associated to $S_\alpha^{2,0,\pm}$, $S_\alpha^{2,N_0,\pm}$, $S_\alpha^{2,2N_0,\pm}$ and $S_\alpha^{2,3N_0,\pm}$.

In order to apply Main Theorem 1 to the hypersurface described above, one would have to apply the balancing arguments as in Sect. 4.3 to deal with the invariant approximate Jacobi fields. Clearly $\Lambda_\theta^\#$ is balanced for each θ because the separation parameters between all hyperspheres are equal and its geodesic segments meet in parallel pairs. Thus it would remain to check only that the derivative of the approximate balancing map has full rank (which corresponds to being invertible in this case because the imposed symmetries rule out all co-kernel coming from induced rotations of $\mathbb{S}^{n+1}$). However, the analysis of the simple example of Sect. 5.1 shows that the kernel of $\mathrm{D}\mathring{B}(0)$ is one-dimensional and consists of the transverse motion $V^k = \sin((\tau + 2\alpha)(N_0 + k)\frac{\partial}{\partial x^1}$ and extended by symmetry. This approximate Jacobi field is induced by the change of the θ-parameter and not by a rotation of Sph^{n+1}. Therefore Main Theorem 1 does not apply to $\Lambda_\theta^\#$ unless $\theta = \pi/2$, in which case there is an additional symmetry (invariance with respect to the rotation $R^{01}_{\pi/2}$) that eliminates this approximate Jacobi field from consideration.

Remark The analogue of the example above in Euclidean space consists of two Delaunay surfaces with non-parallel axes meeting at a common spherical region. It is possible to glue this initial configuration together and perturb it into a CMC hypersurface. This is because the decay conditions at infinity that are built into the function space used in the analysis rules out the approximate Jacobi fields corresponding to the change-of-angle parameter and the translation parameter.

5.3 A related achievable configuration

A modification of the previous example yields an initial configuration of hyperspheres to which Main Theorem 1 does apply. The key is to “freeze” the motion of the θ-parameter without imposing additional symmetries, which can be achieved by adding another set of spheres along the geodesic orthogonal to the initial configuration of Sect. 5.2. The requirement that the spheres at the intersection points of the geodesic with the initial configuration of Sect. 5.2 match perfectly is what freezes the motion in the θ-parameter. That is, choose an integer k_0 and let

$$\Lambda^\# := \left[\bigcup_{k=0}^{N-1} S_\alpha^{2,k,+} \bigcup S_\alpha^{2,k,-}\right] \cup \left[\bigcup_{k=0}^{N-1} S_\alpha^{1,k}\right]$$

where $S_\alpha^{1,k} := (R^{01}_{\tau+2\alpha})^k(S_\alpha)$. Note that $\Lambda^\#$ has the same group of symmetries as before. Its approximate Jacobi fields are those described before as well as the translation of $S_\alpha^{1,k}$ along the geodesic γ_1 for any given $k \in \{1, \dots, N_0 - 1\}$ and extended by symmetry. Again, there are no approximate Jacobi fields associated to the hyperspheres $S_\alpha^{1,0}$, S_α^{1,N_0}, $S_\alpha^{1,2N_0}$ and $S_\alpha^{1,3N_0}$.

The initial configuration $\Lambda^\#$ is balanced because the separation parameters between all hyperspheres are equal and its geodesic segments meet in parallel pairs. Thus to apply Main Theorem 1 it remains to check that the derivative of the approximate balancing map is invertible. Let $T_{1,k} := \dot\gamma_1((\tau + 2\alpha)k)$ and $T_{2,k,\pm} := \left(R^{01}_{\pm(\tau+2\alpha)k_0}\right)_* \dot\gamma_2((\tau + 2\alpha)k)$ be the tangent vectors of the geodesics γ_1 and γ_2 at the centers of the hyperspheres of $\Lambda^\#$ and define

$$\vec{V}^{1,k} := u^k T_{1,k}$$

$$\vec{V}^{2,k,\pm} := v^{k,\pm} T_{2,k,\pm} + w^{k,\pm} \frac{\partial}{\partial x^1}$$

as the displacement parameters of these hyperspheres. Note that

$$u^k = v^{k,\pm} = 0 \qquad k \equiv 0 \mod 4$$

$$w^{0,\pm} = w^{2N_0,\pm} = 0$$

$$u^k = -u^{-k} = -u^{2N_0+k} = u^{2N_0-k} \qquad k = 1, \ldots, N_0 - 1$$

and similarly for v^* and w^* by symmetry. In addition $u^{k_0} = w^{0,+}$ since the corresponding hyperspheres coincide. Thus it is only necessary to analyze the action of $\mathrm{D}\mathring{B}(0)$ on the vector $\vec{V} := (\vec{V}^{1,1}, \ldots, \vec{V}^{1,N_0-1}, \vec{V}^{2,1,+}, \ldots, \vec{V}^{2,N_0-1,+})$ and set $v^k := v^{k,+}$ and $w^k := w^{k,+}$. One finds

$$\mathrm{D}\mathring{B}(0)(\vec{V}) := \begin{pmatrix} \vdots \\ -(n-1)\omega\varepsilon^{n-2}\dot{\varepsilon}\left(2u^k - u^{k+1} - u^{k-1}\right) \\ -(n-1)\omega\varepsilon^{n-2}\dot{\varepsilon}\left(2v^k - v^{k+1} - v^{k-1}\right) \\ -\omega\varepsilon^{n-1}\left(2w^k - w^{k+1} - w^{k-1}\right) \\ \vdots \end{pmatrix}.$$

If the equations in $\mathrm{D}\mathring{B}(0)(\vec{V}) = 0$ were uncoupled, then the kernel would be of the form found in the simple example of Sect. 5.1. If the boundary conditions are included, then it follows that $v^k = 0$ for all k, as well as $u^k = c$ for all k and $w^k = c' \sin((\tau + 2\alpha)(k + N_0))$ for $c, c' \in \mathbb{R}$. But the coupling $2w^0 - 2\sec(\tau + 2\alpha)w^1 = 2u^{k_0} - 2\sec(\tau + 2\alpha)w^1 = 0$ then forces $c = c' = 0$. Hence $\mathrm{D}\mathring{B}(0)$ is invertible and Main Theorem 1 applies to allow $\Lambda^\#$ to be glued together and perturbed into a CMC hypersurface.

5.4 An achievable configuration without any symmetries

The previous example has much less symmetry than the examples constructed in [2] but still possesses a large symmetry group. Further modifications of the ideas of the previous sections leads to examples of initial configurations to which Main Theorem 1 applies with few symmetries or no symmetries at all. These example are naturally quite hard to write down, and in any case the purpose of this final section of the paper is to give the reader the necessary ideas for constructing these examples, so it is sufficient to proceed in the $n = 2$ case.

The first modification leading to a much less symmetric example is to consider $\Lambda^\#$ from Sect. 5.3, except with the new geodesic tilted into the x^3-direction by some angle which is not $\pi/2$. Such an example would still be balanced because its geodesic segments would continue to meet in parallel pairs. Also, such an example would clearly possess no symmetries other than the $x \mapsto -x$ reflection sending a point on $\mathbb{S}^3$ to the antipodal point. However, it is not immediately clear that it is possible to tilt the third geodesic so that equally spaced hyperspheres of radius $\cos(\alpha)$ along the third geodesic line up exactly with the hyperspheres of the same radius along the first two geodesics where these geodesics meet. But a moment's thought reveals that what is needed for some configuration of equally spaced spheres of some radius winding some perhaps large number of times around $\mathbb{S}^3$ to exist is that all the geodesic segments have lengths which are rational multiples of 2π. This, in turn, can be achieved if the three unit vectors N_1, N_2, N_3 orthogonal to the planes containing the three geodesics have $\langle N_i, N_j\rangle \in 2\pi\mathbb{Q}$ for all $i, j \in \{1, 2, 3\}$. This can be achieved. The details of the balancing arguments that prove that Main Theorem 1 applies are identical to the arguments of Sect. 5.3 and thus the configuration above can be glued together and perturbed into a CMC hypersurface.

One final modification of these ideas leads to an example without any symmetries at all. The idea is to perform the same trick of adding in a tilted geodesic to a configuration which does not have the $x \mapsto -x$ antipodal symmetry. Such a configuration is the following: consider three half-geodesics of the form $R^{02}_{2\pi/3}(\gamma_1([0,\pi]))$ and choose a fourth geodesic which is tilted into the x^3-direction. The reader can verify that the fourth geodesic can be chosen in such that equally positioned hyperspheres match appropriately and that the balancing arguments needed to apply Main Theorem 1 hold. Hence this configuration can be glued together and perturbed in a CMC hypersurface as well.

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