Calculus of Variations

Hamiltonian stationary tori in Kähler manifolds

Adrian Butscher · Justin Corvino

Abstract A Hamiltonian stationary Lagrangian submanifold of a Kähler manifold is a Lagrangian submanifold whose volume is stationary under Hamiltonian variations. We find a sufficient condition on the curvature of a Kähler manifold of real dimension four to guarantee the existence of a family of small Hamiltonian stationary Lagrangian tori.

1 Introduction and statement of results

Let M^{2n} be a Kähler manifold with complex structure J, Riemannian metric g, and symplectic form ω. The Lagrangian submanifolds of M are very natural and meaningful objects to consider when M is studied from the symplectic point of view. Upon taking the metric into account, one can study Lagrangian submanifolds which are in some way well-adapted to the metric geometry of M. For instance, Lagrangian submanifolds which are also minimal with respect to the metric g, i.e. which are critical points of the n-dimensional volume functional with respect to compactly supported variations, possess a rich mathematical structure, and their study is an active area of research (see e.g. [6,13]).

It is possible to pose two other natural variational problems amongst Lagrangian submanifolds whose critical points are also mathematically quite interesting. These variational problems are obtained by restricting the class of allowed variations. First, one can demand that the volume of a Lagrangian submanifold Σ is a critical point with respect to only those variations of Σ which preserve the Lagrangian condition; in this case, Σ is said to be *Lagrangian*

Communicated by F. Hélein.

A. Butscher (✉)
Department of Computer Science, Stanford University, Stanford, CA 94305, USA
e-mail: adrian.butscher@gmail.com

J. Corvino
Department of Mathematics, Lafayette College, Easton, PA 18042, USA
e-mail: corvinoj@lafayette.edu

stationary. Since it turns out that a smooth Lagrangian stationary submanifold is necessarily minimal (because the mean curvature vector field of Σ is itself the infinitesimal generator of a Lagrangian variation, as indicated in [14]), points where a Lagrangian stationary submanifold fails to be minimal must be singular points, and what is of interest is the precise nature of the set of singularities. A second variational problem that one can pose is the following. There is a natural sub-class of variations preserving the Lagrangian condition, namely the set of Hamiltonian transformations, which are generated by functions on M; hence one can also demand that the volume of Σ is a critical point with respect to only Hamiltonian variations. In this case, Σ is said to be *Hamiltonian stationary*, and there are indeed examples of non-trivial, smooth, Hamiltonian stationary submanifolds that are not minimal (cf. Lemma 1). In this paper, we focus on the second of these two variational problems.

Hamiltonian stationary submanifolds of a Kähler–Einstein manifold M have been studied by several authors, notably Oh [11,10], Hélein and Romon [3–5], and Schoen and Wolfson [14,15]. Oh initially posed the Lagrangian and Hamiltonian stationary variational problems and derived first and second variation formulæ. Hélein and Romon showed that when M is a Hermitian symmetric space of real dimension four, this stationarity condition can be reformulated as an infinite-dimensional integrable system whose solutions possess a Weierstraß-type representation. Moreover, they found all Hamiltonian stationary, doubly periodic immersions of $\mathbb{R}^2$ into $\mathbb{C}P^2$ using this representation. Finally, Schoen and Wolfson initiated the study of Lagrangian variational problems from the geometric analysis point of view, for the purpose of constructing minimal Lagrangian submanifolds as limits of volume-minimizing sequences of Lagrangian submanifolds.

The approach we take in this paper is to state a general sufficient condition for the existence of a certain type of Hamiltonian stationary submanifold in a Kähler manifold M. Namely, we specify a condition at a point p in M which allows us to construct Hamiltonian stationary tori of sufficiently small radii optimally situated in a neighbourhood of the point p. Of course, a simple motivating example is $\mathbb{C}^n$ where one has the standard tori of any radii built with respect to any chosen unitary frame at any chosen point. These tori will be explicitly used in our construction and will be defined carefully below. But for a more significant example, we note that all *Kähler toric manifolds* contain Hamiltonian stationary Lagrangian tori of the type envisaged here. A Kähler toric manifold is a closed, connected $2n$-dimensional Kähler manifold (M, g, ω, J) equipped with an effective Hamiltonian holomorphic action $\tau : \mathbb{T}^n \to \mathrm{Diff}(M)$ of the standard (real) n-torus $\mathbb{T}^n$. The orbits of the group action turn out to be Hamiltonian stationary Lagrangian submanifolds of M, essentially because the metric g turns out to be equivariant under the action of τ. Furthermore, the image of the *moment map* $\mu_\tau : M \to \mathbb{R}^n$ of τ is a convex polytope P in $\mathbb{R}^n$. Let $M_0 := \mu_\tau^{-1}(int(P))$; then M_0 is an open, dense subset of M that is symplectomorphic to $int(P) \times \mathbb{T}^n$, upon which the action is free. The orbit tori located near the corners of the polytope turn out to have small volume tending to zero at the corners themselves. A discussion of the geometry of Kähler toric manifolds can be found in [1].

On the other hand, in a general Kähler manifold M, one might expect that smooth, small Hamiltonian stationary tori are rather rare, with a condition depending in some way on the ambient geometry of M governing their existence. The archetype for this kind of a result is an analogous construction, due to Ye [16], of small constant mean curvature spheres in a Riemannian manifold M. Ye has shown that it is possible to perturb a sufficiently small geodesic sphere centered at the point $p \in M$ to a hypersurface of constant mean curvature, provided that p is a non-degenerate critical point of the scalar curvature of M.

We now explain and state the Main Theorem to be proved in this paper. Let (M, g, ω, J) be a Kähler manifold, with $\dim_\mathbb{R} M = 4$. Let $U(M)$ denote the unitary frame bundle of M and

choose a point $p \in M$ and a unitary frame $\mathcal{U}_p \in U(M)$ at p. Let (z^1, z^2) be complex normal coordinates for a neighbourhood of p whose coordinate vectors at the origin coincide with $\mathcal{U}_p$. Fix $r := (r_1, r_2) \in \mathbb{R}_+^2$ (the open positive quadrant of $\mathbb{R}^2$) with small $\|r\| = \sqrt{r_1^2 + r_2^2}$, and define the submanifold

$$\Sigma_r(\mathcal{U}_p) := \left\{ \left(r_1 e^{i\theta^1}, r_2 e^{i\theta^2} \right) : (\theta^1, \theta^2) \in \mathbb{T}^2 \right\}.$$

If M were $\mathbb{C}^2$, then $\Sigma_r(\mathcal{U}_p)$ would be Hamiltonian stationary Lagrangian for all r and $\mathcal{U}_p$. In general, $\Sigma_r(\mathcal{U}_p)$ is almost Hamiltonian stationary Lagrangian when $\|r\|$ is very small, as the ambient metric is nearly Euclidean in complex normal coordinates. We express the tangent vectors $\frac{\partial}{\partial \theta^k}$ in complex notation as $\frac{\partial}{\partial \theta^k} = i r_k e^{i\theta^k} \frac{\partial}{\partial z^k}$. Given a section X of the bundle $J(T\Sigma_r(\mathcal{U}_p))$, we write $X = \sum_{k=1}^2 X^k J(\frac{\partial}{\partial \theta^k}) = \sum_{k=1}^2 -r_k X^k e^{i\theta^k} \frac{\partial}{\partial z^k}$, and define the deformed submanifold

$$\mu_X\left(\Sigma_r(\mathcal{U}_p)\right) := \left\{ \left(r_1(1 - X^1(\theta)) e^{i\theta^1}, r_2(1 - X^2(\theta)) e^{i\theta^2} \right) : (\theta^1, \theta^2) \in \mathbb{T}^2 \right\}.$$

The condition that the deformed submanifold be a Hamiltonian stationary Lagrangian submanifold can now be phrased as a partial differential equation for the functions X^1 and X^2. It will be shown in this paper that these equations can be solved subject to an existence condition that can be expressed in terms of the complex curvature tensor of M.

In order to phrase the existence condition, we now define the relevant curvature functions on unitary frames. Let $R^{\mathbb{C}}_{I\bar{J}K\bar{L}} := R^{\mathbb{C}}\left(\frac{\partial}{\partial z^I}, \frac{\partial}{\partial \bar{z}^J}, \frac{\partial}{\partial z^K}, \frac{\partial}{\partial \bar{z}^L} \right)$ be the components of the complexified Riemann curvature tensor, and denote the components of the complex covariant derivatives of this tensor by $(DR^{\mathbb{C}})_{I\bar{J}K\bar{L}\bar{M}} := R^{\mathbb{C}}_{I\bar{J}K\bar{L};\bar{M}}$. Now given any frame $\mathcal{U}_p \in U(M)$ we can define

$$\mathcal{G}_r(\mathcal{U}_p) := i \begin{pmatrix} r_1^2 R^{\mathbb{C}}_{1\bar{1}1\bar{1};\bar{1}} + r_2^2 R^{\mathbb{C}}_{2\bar{2}2\bar{2};\bar{1}} \\ r_1^2 R^{\mathbb{C}}_{1\bar{1}1\bar{1};\bar{2}} + r_2^2 R^{\mathbb{C}}_{2\bar{2}2\bar{2};\bar{2}} \\ 2r_2^2 R^{\mathbb{C}}_{2\bar{2}2\bar{1}} - 2r_1^2 R^{\mathbb{C}}_{1\bar{1}2\bar{1}} \end{pmatrix} \in \mathbb{C}^3 \approx \mathbb{R}^6. \tag{1}$$

Observe that the unitary group acts on $U(M)$ by matrix multiplication in the fiber direction. The subgroup $\mathbf{\Delta} \subset U(2)$ of unitary diagonal matrices thus acts on $U(M)$ as well, and furthermore, $\Sigma_r(\mathcal{U}_p) = \Sigma_r(D \cdot \mathcal{U}_p)$ for all diagonal matrices $D \in \mathbf{\Delta}$. The existence condition on $(\mathcal{U}_p, r)$ that we will use is that the curvature function $\mathcal{G}_r$ has a non-degenerate zero at $\mathcal{U}_p$, along directions transverse to the orbit of $\mathcal{U}_p$ under $\mathbf{\Delta}$: we say such a frame is $\mathbf{\Delta}$-*non-degenerate*. By the invariance of $R^{\mathbb{C}}$ under the action of $\mathbf{\Delta}$, we can define a function $\mathcal{F}_r : U(M)/\mathbf{\Delta} \to \mathbb{R}$ by $\mathcal{F}_r([\mathcal{U}_p]) = r_1^2 R^{\mathbb{C}}_{1\bar{1}1\bar{1}} + r_2^2 R^{\mathbb{C}}_{2\bar{2}2\bar{2}}$. We remark that the mapping $\mathcal{G}_r$ is obtained precisely by applying $D\mathcal{F}_r$ to a basis of the tangent space of $U(M)/\mathbf{\Delta}$. The top two complex components come from the spatial derivatives, and the last component comes from differentiating in the frame directions. Thus $\mathcal{U}_p$ is a $\mathbf{\Delta}$-non-degenerate zero for $\mathcal{G}_r$ precisely when $\mathcal{F}_r$ has a non-degenerate critical point at $[\mathcal{U}_p]$. Note that this condition is preserved when the vector r is scaled by a constant multiple.

Main Theorem *Let (M, g, ω, J) be a Kähler manifold, with $\dim_{\mathbb{R}} M = 4$. Let $r := (r_1, r_2) = \|r\|\hat{r} \in \mathbb{R}_+^2$ where $\|\hat{r}\| = 1$, and suppose $\mathcal{U}_p \in U(M)$ is such that $\mathcal{U}_p \in U(M)$ is a $\mathbf{\Delta}$-non-degenerate zero for $\mathcal{G}_{\hat{r}}$. Then for $\|r\|$ sufficiently small, there exists $\mathcal{U}_{p'} \in U(M)$ and a section $X \in \Gamma(J(T\Sigma_r(\mathcal{U}_{p'})))$ so that the submanifold $\mu_X(\Sigma_r(\mathcal{U}_{p'}))$ is smooth and Hamiltonian stationary Lagrangian. Moreover, for any $\alpha \in (0, 1)$, we have $\|X\|_{0,\Sigma_r} + \|r\|\|\nabla X\|_{0,\Sigma_r} + \|r\|^2\|\nabla^2 X\|_{0,\Sigma_r} + \|r\|^{2+\alpha} \left[\nabla^2 u \right]_{\alpha, \Sigma_r} = \mathcal{O}(\|r\|^3)$, while the*

distance between $\mathcal{U}_p$ *and* $\mathcal{U}_{p'}$ *as points in* $U(M)$ *is* $\mathcal{O}(\|r\|)$, *and the distance between* p *and* p' *is* $O(\|r\|^2)$.

We note as a direct corollary that it is possible to extend the Main Theorem slightly in order to answer a more general question. That is, the Main Theorem finds a Hamiltonian stationary submanifold that is a small perturbation of Σ_r for $\|r\|$ sufficiently small. Now one can ask if it is possible to find neighbouring Hamiltonian stationary Lagrangian submanifolds which are perturbations of $\Sigma_{r'}$ with r' sufficiently close to r. The answer to this question is that one can indeed find such submanifolds because the **Δ**-non-degenerate zeros of the family of functionals $\mathcal{G}_{r'}$ with r' varying in a neighbourhood of r are *stable*. That is, if r' is sufficiently close to r, then $\mathcal{G}_{r'}$ has a Δ-non-degenerate zero $\mathcal{U}_{p(r')}$ near $\mathcal{U}_p$. By the Implicit Function Theorem, moreover, one can arrange the association $r' \mapsto \mathcal{U}_{p(r')}$ to be smooth.

Corollary *Let* $r := (r_1, r_2) \in \mathbb{R}_+^2$ *with* $\|r\|$ *sufficiently small, and suppose* $\mathcal{U}_p \in U(M)$ *is a* **Δ***-non-degenerate zero for* $\mathcal{G}_r$. *Then one can find a neighbourhood* $\mathcal{O} \subset \mathbb{R}_+^2$ *containing* r *so that for each* $r' \in \mathcal{O}$, $\Sigma_{r'}(\mathcal{U}_p)$ *can be perturbed into a Hamiltonian stationary Lagrangian submanifold of* M, *the family of which is smoothly parametrized by* r'.

The Main Theorem will be proved following broadly similar lines as the proof of Ye's result. That is, for each $\mathcal{U}_p$ and sufficiently small $\|r\|$, a section X will be found so that $\mu_X(\Sigma_r(\mathcal{U}_p))$ is *almost* Lagrangian and Hamiltonian stationary; in fact the small error will be arranged to lie in a certain finite-dimensional space. The discrepancy comes from the fact that the Hamiltonian stationary differential operator possesses an *approximate co-kernel*: the linearized Hamiltonian stationary differential operator can be expanded in powers of $\|r\|$, where the lowest-order term is the linearized Hamiltonian stationary differential operator of $\mathbb{C}^2$, whose co-kernel contains a six-dimensional space arising from translation and $U(2)$-rotation. This discrepancy constitutes an obstruction to solvability. Only when $\Sigma_r(\mathcal{U}_p)$ is very special (such that the image of the Hamiltonian stationary differential operator acting on $\Sigma_r(\mathcal{U}_p)$ is orthogonal to the associated co-kernel to lowest order in $\|r\|$) can a solution be found. This special situation arises exactly when $\mathcal{G}_r(\mathcal{U}_p) = 0$. The perturbation $\mu_X(\Sigma_r(\mathcal{U}_p))$ now produces a Hamiltonian stationary Lagrangian submanifold up to an even smaller error term—and this remaining error term can be corrected as well when the non-degeneracy condition holds.

The existence condition described above is qualitatively similar to Ye's condition in that it involves the ambient curvature tensor of M. But of course the condition here takes into account the freedom to choose the complex frame with respect to which $\Sigma_r(\mathcal{U}_p)$ is built as well as the point p about which $\Sigma_r(\mathcal{U}_p)$ is located. As with Ye's condition, it is not always the case that there exists $\mathcal{U}_p$ satisfying the non-degeneracy condition. For example, this occurs in the case of $\mathbb{C}P^2$ and of $\mathbb{C}^2$, despite the fact that both spaces contain small Hamiltonian stationary Lagrangian tori. These examples can be seen as analogues of situation in $\mathbb{R}^n$, a space which fails to satisfy the non-degeneracy criterion of Ye and where constant mean curvature spheres nevertheless come in great abundance. The existence of a non-degenerate frame $\mathcal{U}_p$ can also fail to hold in Kähler manifolds with one-parameter families of isometries. Such examples also occur in the study of constant mean curvature spheres, and it should be noted that Pacard and Xu have recently strengthened Ye's result to handle these cases. They did this by analyzing the second-to-lowest-order term in the expansion of the linearized constant mean curvature operator and replacing Ye's non-degeneracy condition with a different condition that can be applied even when Ye's original condition cannot. They can deduce from their condition that *every* compact Riemannian manifold must have at least one point p for which sufficiently small geodesic spheres centered at p can be perturbed to hypersurfaces of

constant mean curvature [12]. A similar strengthening should be possible in the Hamiltonian stationary Lagrangian case as well.

Recent work by Joyce, Lee and Schoen [7] establishes in a very general sense the existence of Hamiltonian stationary Lagrangian submanifolds in a symplectic manifold M. The submanifolds constructed are modeled on certain Hamiltonian stationary Lagrangian submanifolds of $\mathbb{C}^n$, the class of which contains the tori we use in the present work. They embed these into M using adapted Darboux coordinates, where they are Lagrangian but only approximately Hamiltonian stationary. These authors then establish that it is always possible (when M is compact) to find an exactly Hamiltonian stationary Lagrangian submanifold near at least one of their approximate solutions. This is because the existence condition that must be satisfied is that a certain functional on $U(M)$ has a critical point (and since $U(M)$ is compact when M is compact, the existence condition must hold for some $\mathcal{U}_p \in U(M)$).

Though some of the analysis in [7] is similar to ours, there is a key difference that we note here. In [7], the authors employ adapted Darboux coordinates so that they are able to embed their model submanifolds as Lagrangian submanifolds in M. This allows them to find a nearby Hamiltonian stationary Lagrangian by solving a single, scalar differential equation. We employ complex normal coordinates in which the metric and symplectic form of M are perturbations of the metric and symplectic form of $\mathbb{C}^n$ while the complex structure is everywhere standard. Embedding $\Sigma_r(\mathcal{U}_p)$ into this coordinate chart results in a submanifold which is approximately Lagrangian and approximately Hamiltonian stationary. We thus have to include the Lagrangian condition as part of the system of partial differential equations to analyze. The reason that we do this is to exploit the fact that the metric and symplectic form of M in this coordinate chart can be expanded in terms of the ambient curvature tensor, and hence we can formulate our existence condition very concretely in terms of the curvature. Since the time our work was first announced, Lee [9] was able to establish the existence of suitable Darboux coordinates to yield such an existence result following the approach of [7]. As remarked in [7], the curvature condition in a preprint version of this work differed from that in [9]; this was due a to sign error that has been fixed in the present work, thus giving two independent calculations of the non-degeneracy condition.

In the case of constant mean curvature spheres, Ye's existence condition is essentially that the sphere for which the perturbation argument succeeds is the one for which the area functional, subject to the constraint of constant enclosed volume, and restricted to the space of embedded geodesic spheres, has a non-degenerate critical point. Ye then deduces the non-degeneracy of the scalar curvature from this by expanding the constrained, restricted area functional in terms of the background geometry of M. In the case of the Hamiltonian stationary Lagrangian submanifolds studied by Joyce, Lee and Schoen, their existence condition amounts to having a critical point for the volume functional restricted to the space of approximate solutions, cf. [7,9]. In our case, we make a simple *Ansatz* for solutions, which moves us off the constraint set of Lagrangian submanifolds, but we can arrange approximate solutions with small enough errors so that in the end we are able to project onto the space of Lagrangian submanifolds to a solution of the Hamiltonian stationary equation.

2 Geometric preliminaries

2.1 Kähler manifolds

A complex manifold M of real dimension $2n$ and integrable complex structure J is said to be *Kähler* if it possesses a Riemannian metric g for which J is an isometry, as well as a

symplectic form ω satisfying the compatibility condition $\omega(X, Y) = g(JX, Y)$ for all tangent vectors X, Y. We recall that the complexification $T_{\mathbb{C}}M$ of the real tangent bundle TM splits: $T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$. The map $TM \hookrightarrow T_{\mathbb{C}}M \to T^{(1,0)}M$, defined by inclusion followed by projection, induces an isomorphism between T_pM and $T_p^{(1,0)}M$, which gives us a way to encode T_pM in complex notation. In local coordinates $z^k = x^k + \mathrm{i}y^k$, we have

$$\frac{\partial}{\partial x^k} = \frac{\partial}{\partial z^k} + \frac{\partial}{\partial \bar{z}^k}, \quad \frac{\partial}{\partial y^k} = \mathrm{i}\frac{\partial}{\partial z^k} - \mathrm{i}\frac{\partial}{\partial \bar{z}^k},$$

so that under this isomorphism, $\frac{\partial}{\partial x^k}$ corresponds to $\frac{\partial}{\partial z^k}$, and $\frac{\partial}{\partial y^k}$ to $\mathrm{i}\frac{\partial}{\partial z^k}$. Any vector $X \in T_pM$ thus has the form $X = V + \bar{V}$, for some $V \in T_p^{(1,0)}M$; we let $X_{\mathbb{C}} = V$. It is not hard to show that if X and Y are smooth sections of TM, so that $Y_{\mathbb{C}}$ is holomorphic, then the Levi-Civita connection D satisfies $(D_X Y)_{\mathbb{C}} = D_{X_{\mathbb{C}}} Y_{\mathbb{C}}$. Thus we may blur the distinction between X and $X_{\mathbb{C}}$ when using complex notation. Standard references for Kähler manifolds are [2] and [8]. What follows is a brief description, for the purpose of fixing terminology and notation, of those aspects of Kähler geometry that will be relevant for what follows.

The question of interest is the nature of the *local* geometry of a Kähler manifold. In a general Kähler manifold, it is always possible to find local complex coordinates for a neighbourhood $\mathcal{V}$ of any point $p \in M$, and a function $F : \mathcal{V} \to \mathbb{R}$, called the *Kähler potential*, so that the metric and symplectic form are:

$$g = 2\mathbf{Re}\sum_{k,\ell}\left(\frac{\partial^2 F}{\partial z^k \partial \bar{z}^\ell}\mathrm{d}z^k \otimes \mathrm{d}\bar{z}^\ell\right) = \frac{1}{2}\sum_{k,\ell}\left(\frac{\partial^2 F}{\partial x^k \partial x^\ell} + \frac{\partial^2 F}{\partial y^k \partial y^\ell}\right)\left(\mathrm{d}x^k \otimes \mathrm{d}x^\ell + \mathrm{d}y^k \otimes \mathrm{d}y^\ell\right)$$

$$+ \frac{1}{2}\sum_{k,\ell}\left(\frac{\partial^2 F}{\partial y^k \partial x^\ell} - \frac{\partial^2 F}{\partial x^k \partial y^\ell}\right)\left(\mathrm{d}y^k \otimes \mathrm{d}x^\ell - \mathrm{d}x^k \otimes \mathrm{d}y^\ell\right),$$

$$\omega = -2\,\mathbf{Im}\sum_{k,\ell}\left(\frac{\partial^2 F}{\partial z^k \partial \bar{z}^\ell}\mathrm{d}z^k \otimes \mathrm{d}\bar{z}^\ell\right) = \frac{1}{2}\sum_{k,\ell}\left(\frac{\partial^2 F}{\partial x^k \partial x^\ell} + \frac{\partial^2 F}{\partial y^k \partial y^\ell}\right)\left(\mathrm{d}x^k \otimes \mathrm{d}y^\ell - \mathrm{d}y^k \otimes \mathrm{d}x^\ell\right)$$

$$+ \frac{1}{2}\sum_{k,\ell}\left(\frac{\partial^2 F}{\partial y^k \partial x^\ell} - \frac{\partial^2 F}{\partial x^k \partial y^\ell}\right)\left(\mathrm{d}x^k \otimes \mathrm{d}x^\ell + \mathrm{d}y^k \otimes \mathrm{d}y^\ell\right),$$

in local complex coordinates $(z^1, \ldots, z^n)$ or local real coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ for $\mathcal{V}$, which are related by $z^k = x^k + \mathrm{i}y^k$. Note that

$$\omega = \frac{1}{2}\sum_k \mathrm{d}\left(\frac{\partial F}{\partial x^k}\mathrm{d}y^k - \frac{\partial F}{\partial y^k}\mathrm{d}x^k\right),$$

which is consistent with the fact that $\mathrm{d}\omega = 0$, and locally, closed forms are exact. Write $\omega := \mathrm{d}\alpha$, where α is called the *Liouville form* of ω, and write $\mathring{\alpha} := \frac{1}{2}\sum_k\left(x^k\mathrm{d}y^k - y^k\mathrm{d}x^k\right)$ for the Liouville form of the standard symplectic form. Note also that the Kähler potential is unique up to the addition of a function φ satisfying $\frac{\partial^2 \varphi}{\partial z^k \partial \bar{z}^\ell} = 0$ for all k, ℓ.

Consider the simplest example of a Kähler manifold: this is $\mathbb{C}^n$ equipped with the standard Euclidean metric $\mathring{g} := \mathbf{Re}\left(\sum_k \mathrm{d}z^k \otimes \mathrm{d}\bar{z}^k\right)$ and the standard symplectic form $\mathring{\omega} := -\mathbf{Im}\left(\sum_k \mathrm{d}z^k \otimes \mathrm{d}\bar{z}^k\right)$ (both given in complex coordinates), as well as the standard complex structure (which coincides with multiplication by $\sqrt{-1}$ in complex coordinates). In a Kähler manifold, it is always possible to find local complex coordinates for a neighbourhood $\mathcal{V}$ of any point $p \in M$ in which the complex structure is standard everywhere in $\mathcal{V}$, and the metric and symplectic form are standard at p with vanishing derivatives. Indeed, one can

additionally show that it is possible to choose F near p having the following expression in coordinates (where $z = 0$ corresponds to $p \in M$)

$$F(z, \bar{z}) := \frac{1}{2}\|z\|^2 + \hat{F}(z, \bar{z}) \tag{2}$$

where $\hat{F}$ vanishes at least to order four in z and $\bar{z}$. Hence $\frac{\partial^2 F}{\partial z^k \partial \bar{z}^\ell} = \frac{1}{2}\delta_{k\ell} + \mathcal{O}(\|z\|^2)$. Consequently, the Kähler structures near the origin are perturbations of the standard structures $\mathring{g}$ and $\mathring{\omega}$, whose Kähler potential is $\mathring{F}(z, \bar{z}) := \frac{1}{2}\|z\|^2$. Moreover, this form for the potential, and hence for the metric and symplectic form, is $U(n)$-invariant.

The complexified curvature tensor of a Kähler manifold in local coordinates in $\mathcal{V}$ can be expressed in terms of the Kähler potential: the complexified curvature tensor $R^{\mathbb{C}}$ satisfies [8]

$$R^{\mathbb{C}}_{K\bar{L}M\bar{N}} = \frac{\partial^4 F}{\partial z^K \partial \bar{z}^L \partial z^M \partial \bar{z}^N} - \sum_{U,V} g^{\bar{U}V} \frac{\partial^3 F}{\partial z^K \partial \bar{z}^U \partial z^M} \frac{\partial^3 F}{\partial \bar{z}^L \partial z^V \partial \bar{z}^N}.$$

For coordinates in which the Kähler potential is of the form (2), we have $\partial^3 F(0) = 0$, so that (where we note we use the convention that a comma denotes a partial derivative, while a semicolon denotes a covariant derivative)

$$R^{\mathbb{C}}_{K\bar{L}M\bar{N}}(p) = \frac{\partial^4 F(0)}{\partial z^K \partial \bar{z}^L \partial z^M \partial \bar{z}^N} \quad \text{and} \quad R^{\mathbb{C}}_{K\bar{L}M\bar{N};\bar{S}}(p) = \frac{\partial^5 F(0)}{\partial z^K \partial \bar{z}^L \partial z^M \partial \bar{z}^N \partial \bar{z}^S}. \tag{3}$$

This can be expressed in terms of the usual Riemann curvature tensor R by realizing that

$$R^{\mathbb{C}}_{K\bar{L}M\bar{N}}(p) = R\Big|_p \left(\frac{\partial}{\partial z^K}, \frac{\partial}{\partial \bar{z}^L}, \frac{\partial}{\partial z^M}, \frac{\partial}{\partial \bar{z}^N} \right)$$

$$R^{\mathbb{C}}_{K\bar{L}M\bar{N};\bar{S}}(p) = \frac{\partial}{\partial \bar{z}^S}\Big|_p R \left(\frac{\partial}{\partial z^K}, \frac{\partial}{\partial \bar{z}^L}, \frac{\partial}{\partial z^M}, \frac{\partial}{\partial \bar{z}^N} \right)$$

at the point p, setting $\frac{\partial}{\partial z^K} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - \mathrm{i}\frac{\partial}{\partial y^k} \right)$, and similarly for the conjugate. Consequently

$$F_{,K\bar{L}M\bar{N}}(0) = \frac{1}{4} \left((R_{k\ell mn} - R_{k\bar{\ell}m\bar{n}}) + \mathrm{i}\left(R_{k\ell m\bar{n}} + R_{k\bar{\ell}mn} \right) \right)\Big|_p$$

$$F_{,K\bar{L}M\bar{N}\bar{S}}(0) = \frac{1}{8} \left((R_{k\ell mn;s} - R_{k\bar{\ell}m\bar{n};s} - R_{k\ell m\bar{n};\bar{s}} - R_{k\bar{\ell}mn;\bar{s}}) \right.$$
$$\left. + \mathrm{i}\left(R_{k\ell mn;\bar{s}} - R_{k\bar{\ell}m\bar{n};\bar{s}} + R_{k\ell m\bar{n};s} + R_{k\bar{\ell}mn;s} \right) \right)\Big|_p$$

where an un-barred, lower-case index refers to a coordinate vector of the form $\frac{\partial}{\partial x^k}$, while a barred, lower-case index refers to a coordinate vector of the form $\frac{\partial}{\partial y^k}$.

2.2 Hamiltonian stationary Lagrangian submanifolds

Interesting submanifolds of a Kähler manifold can be characterized by the effect of the action of J on tangent spaces. For instance, a *complex* submanifold of M^{2n} is one whose tangent spaces are invariant under J. Two classes of submanifolds of importance in this paper are defined in terms of a complementary condition to that of a complex submanifold. An n-dimensional submanifold Σ is called *Lagrangian* if $J(T_p\Sigma)$ is orthogonal to $T_p\Sigma$ for each $p \in \Sigma$. Hence a Lagrangian submanifold satisfies $\omega(X, Y) = 0$ for all $X, Y \in T_p\Sigma$ and

$p \in \Sigma$. More generally, an n-dimensional submanifold Σ for which $J(T_p\Sigma)$ is transverse to $T_p\Sigma$ for each $p \in \Sigma$ is called *totally real*.

We will be interested in diffeomorphisms of M that preserve some or all aspects of its Kähler structure. The diffeomorphisms which preserve the full Kähler structure are the *holomorphic isometries* and are quite rare in general. In $\mathbb{C}^n$, though, there are holomorphic isometries: these are the $U(n)$-rotations and translations. The diffeomorphisms which preserve the symplectic form but not necessarily the metric are called *symplectomorphisms*. Every Kähler manifold possesses symplectomorphisms; indeed, for each function $u : M \to \mathbb{R}$ the one-parameter family of diffeomorphisms obtained by integrating the vector field X defined by $X \lrcorner \omega := du$ are symplectomorphisms. These diffeomorphisms are called *Hamiltonian*. The condition of being totally real or Lagrangian is preserved by symplectomorphisms.

Consider now a Lagrangian submanifold $\Sigma \subset M$. If Σ is a critical point of the n-dimensional volume functional amongst *all* possible compactly supported variations, then Σ is *minimal*, in which case the mean curvature vector $\vec{H}_\Sigma$ of Σ vanishes. Suppose, however, that Σ is merely a critical point of the n-dimensional volume amongst only Hamiltonian variations, and thus is Hamiltonian stationary Lagrangian. By computing the Euler-Lagrange equations for Σ, it becomes clear that being Hamiltonian stationary is in general a strictly weaker condition than being minimal. Indeed, let ϕ_t be a one-parameter family of Hamiltonian diffeomorphisms of M with infinitesimal deformation vector field X satisfying $X \lrcorner \omega = du$ for $u : M \to \mathbb{R}$. Then

$$
\begin{aligned}
0 = \left. \frac{d}{dt} \operatorname{Vol}(\phi_t(\Sigma)) \right|_{t=0} &= - \int_\Sigma g(\vec{H}_\Sigma, X) d\operatorname{Vol}_\Sigma \\
&= - \int_\Sigma \omega(X, J\vec{H}_\Sigma) d\operatorname{Vol}_\Sigma \\
&= - \int_\Sigma du(J\vec{H}_\Sigma) d\operatorname{Vol}_\Sigma \\
&= \int_\Sigma u \, \nabla \cdot \left(J\vec{H}_\Sigma \right) d\operatorname{Vol}_\Sigma \tag{4}
\end{aligned}
$$

by Stokes' Theorem. We let D be the connection associated with the ambient metric g, while ∇ is the induced connection of Σ, and $\nabla \cdot$ is the divergence operator. Since (4) must hold for all functions u, it must be the case that the mean curvature of Σ satisfies

$$
\nabla \cdot \left(J\vec{H}_\Sigma \right) = 0. \tag{5}
$$

Equation 5 will be solved in this paper to find Hamiltonian stationary Lagrangian submanifolds.

Observe that since Σ is Lagrangian and $\vec{H}_\Sigma$ is normal to Σ, then $J\vec{H}_\Sigma$ is tangent to Σ and taking its divergence with respect to the induced connection makes sense. It is convenient to introduce some notation at this point so that the mean curvature (and second fundamental form) of a totally real submanifold can be treated in a similar manner. To this end, let Σ be totally real and define the *symplectic second fundamental form* and the *symplectic mean curvature* of Σ by the formulæ

$$
B(X, Y, Z) := \omega\left((D_X Y)^\perp, Z \right) \quad \text{and} \quad H(Z) := \operatorname{Trace}(B(\cdot, \cdot, Z))
$$

where $W^\perp$ is the orthogonal projection of a vector $W = W^\parallel + W^\perp$ defined at a point $p \in \Sigma$ to the normal bundle of Σ at p. The symplectic mean curvature is thus a one-form on Σ, and so the identity

$$H(Z) = \omega(\vec{H}_\Sigma, Z) = g(J\vec{H}_\Sigma, Z)$$

shows that H is dual to $(J\vec{H}_\Sigma)^\parallel$. In the Lagrangian setting, then, Eq. 5 can be written $\nabla \cdot H = 0$. We will sometimes let $H(\Sigma) := H$ to mark the dependence on Σ.

Remark The following observation about the symplectic second fundamental form is important. If Σ is Lagrangian then $B(X, Y, Z) = \omega(D_X Y, Z)$ for all vector fields X, Y, Z tangent to Σ since $\omega((D_X Y)^\parallel, Z) = 0$. Hence we have the usual symmetry $B(X, Y, Z) = B(Y, X, Z)$. In addition, we have $B(X, Z, Y) = g(JD_X Z, Y) = g(D_X JZ, Y) = -g(JZ, D_X Y) = g(JD_X Y, Z) = B(X, Y, Z)$. Consequently the symplectic fundamental form of a Lagrangian submanifold is fully symmetric in all of its slots.

3 Constructing the approximate solution

Let us assume from now on (unless indicated otherwise) that the real dimension of the ambient manifold is four, and thus that the dimension of a Hamiltonian stationary Lagrangian submanifold is two, since this simplifies the presentation of the results and their proofs. Many of the forthcoming calculations are written for any dimension, while others can be generalized to higher dimensions, and similar results will hold (cf. [7,9]).

3.1 Rescaling the ambient manifold

Choose a point $p \in M$ and find local complex coordinates so that a small neighbourhood $\mathcal{V}$ of p maps to a small neighbourhood $\mathcal{V}_0$ of the origin in $\mathbb{C}^2$. Moreover, let these coordinates be such that the metric and symplectic form are of the type discussed in Sect. 2.1. Assume that the diameter of this neighbourhood is $\rho_0 \in (0, 1)$; let $r = (r_1, r_2)$, with $\|r\| < \rho_0$, be the radii of the Hamiltonian stationary Lagrangian torus on which our construction will be based, and set $\rho := \|r\|$. Now consider coordinates $\varphi_\rho : \rho^{-1}\mathcal{V}_0 \to \mathcal{V}$ obtained by rescaling the given coordinates by $z \mapsto \rho z$, and also re-scale the metric and symplectic form via

$$g_\rho := \rho^{-2}\varphi_\rho^* g, \quad \text{and} \quad \omega_\rho := \rho^{-2}\varphi_\rho^* \omega. \tag{6}$$

As a result, we obtain a new Kähler metric g_ρ on a large neighbourhood $\rho^{-1}\mathcal{V}_0 = \|r\|^{-1}\mathcal{V}_0$ of the origin in $\mathbb{C}^2$, where the complex structure is standard and the Kähler potential is

$$F_\rho(z, \bar{z}) := \frac{1}{2}\|z\|^2 + \rho^2 \hat{F}_\rho(z, \bar{z}) \tag{7}$$

with $\hat{F}_\rho(z, \bar{z}) := \rho^{-4}\hat{F}(\rho z, \rho\bar{z})$. Furthermore, the Hamiltonian stationary Lagrangian condition is unchanged under this re-scaling, and the torus $\rho^{-1}\Sigma_r = \Sigma_{\hat{r}}$ has *unit* radius vector $\hat{r} = (\hat{r}_1, \hat{r}_2)$. Therefore, in order to construct a Hamiltonian stationary Lagrangian torus of small radii near p, it is sufficient to construct a Hamiltonian stationary Lagrangian torus with unit radius vector near the origin in $\mathbb{C}^2$ with Kähler potential F_ρ, with ρ sufficiently small. We note that as $\rho \to 0^+$, (g_ρ, ω_ρ) converges smoothly on the unit ball $\mathcal{B}$ to the standard structure $(\mathring{g}, \mathring{\omega})$ (cf. Lemma 10).

Remark The advantage of working with these scaled coordinates is that it is now possible to express the deviation of the background geometry from Euclidean space very efficiently

using the parameter ρ. In particular, $\rho^2 \hat{F}_\rho(z, \bar{z}) = F_\rho(z, \bar{z}) - \frac{1}{2}\|z\|^2$ has a Taylor expansion in z and $\bar{z}$ starting at order four, each (ρ-dependent) coefficient of which is $\mathcal{O}(\rho^2)$. Note also that the standard Hölder norms re-scale in a natural way.

3.2 The approximate solution

Let $U(M)$ denote the unitary frame bundle of M and choose a point $p \in M$ and a unitary frame $\mathcal{U}_p \in U(M)$ at p. Let (z^1, z^2) be complex normal coordinates for a neighbourhood of p whose coordinate vectors at the origin coincide with $\mathcal{U}_p$. Now let $r := (r_1, r_2)$ be some fixed vector belonging to $\mathbb{R}^2_+$, the open positive quadrant of $\mathbb{R}^2$, with $\|r\| = 1$. Define the two-dimensional submanifold of $\mathbb{C}^2$ given by

$$\Sigma_r(\mathcal{U}_p) := \left\{ \left(r_1 e^{i\theta^1}, r_2 e^{i\theta^2} \right) : (\theta^1, \theta^2) \in \mathbb{T}^2 \right\}.$$

Note that $\Sigma_r(\mathcal{U}_p)$ is the image of the $\mathbb{T}^2$ under the embedding $\mu_0 : (\theta^1, \theta^2) \mapsto \left(r_1 e^{i\theta^1}, r_2 e^{i\theta^2} \right)$. We will denote $\Sigma_r := \Sigma_r(\mathcal{U}_p)$ when it is not necessary to speak explicitly of the frame $\mathcal{U}_p$ from which $\Sigma_r(\mathcal{U}_p)$ is built.

The following result motivates the use of Σ_r as an *approximate solution* of the problem of finding Hamiltonian stationary Lagrangian submanifolds in arbitrary Kähler manifolds.

Lemma 1 *The submanifold Σ_r is Hamiltonian stationary Lagrangian with respect to the standard Kähler structure $(\mathring{g}, \mathring{\omega}, J)$ of $\mathbb{C}^2$. In fact, the symplectic second fundamental form $\mathring{B}$ and the symplectic mean curvature $\mathring{H}$ are parallel.*

Proof We include this standard calculation for the convenience of the reader. To begin, the tangent vectors of Σ_r can be found by differentiating in θ. In complex notation, these are $E_k := i r_k e^{i\theta^k} \frac{\partial}{\partial z^k}$, for $k = 1, 2$. From this we can immediately compute the components of the induced metric $\mathring{h}$ and those of $\mathring{\omega}$ restricted to Σ_r. Indeed, since the Kähler potential is $\mathring{F}(z, \bar{z}) = \frac{1}{2}\|z\|^2$, we can read off the induced metric and pullback of the symplectic form in terms of the real and imaginary parts, respectively, of

$$\sum_s dz^s \otimes d\bar{z}^s (E_k, \bar{E}_\ell) = \sum_s r_k r_\ell e^{i\theta^k} \delta_{sk}(-i e^{-i\theta^\ell} \delta_{s\ell}) = r_k^2 \delta_{k\ell}.$$

Thus $\mathring{\omega}$ vanishes on Σ_r, and so Σ_r is Lagrangian. The induced metric is given by $\mathring{h}_{k\ell} = r_k^2 \delta_{k\ell}$.

Let the ambient connection be $\mathring{D}$. The covariant derivatives of the tangent vector fields of the embedding with respect to $\mathring{g}$ in complex notation, are

$$\mathring{D}_{E_k} E_\ell = \frac{\partial}{\partial \theta^k} (i r_\ell e^{i\theta^\ell}) \frac{\partial}{\partial z^\ell} = -r_\ell \delta_{k\ell} e^{i\theta^\ell} \frac{\partial}{\partial z^\ell} = \delta_{k\ell} J E_\ell.$$

Since Σ_r is Lagrangian, we therefore see that the parallel part $(\mathring{D}_{E_k} E_\ell)^\|$ vanishes. We can now compute the symplectic second fundamental form. That is,

$$\begin{aligned}
\mathring{B}_{k\ell j} &= \mathring{\omega}(\mathring{D}_{E_k} E_\ell - (\mathring{D}_{E_k} E_\ell)^\|, E_j) = \mathring{\omega}(\mathring{D}_{E_k} E_\ell, E_j) \\
&= -\mathbf{Im} \sum_s dz^s \otimes d\bar{z}^s \left(\mathring{D}_{E_k} E_\ell, \bar{E}_j \right) \\
&= -\mathbf{Im} \sum_s dz^s \otimes d\bar{z}^s \left(-r_\ell \delta_{k\ell} e^{i\theta^\ell} \frac{\partial}{\partial z^\ell}, -i r_j e^{-i\theta^j} \frac{\partial}{\partial \bar{z}^j} \right) \\
&= -r_m^2 \delta_{km} \delta_{\ell m} \delta_{jm},
\end{aligned}$$

where m can be any of k, ℓ or j. This emphasizes the symmetry of $\mathring{B}$ in its indices, as proved more generally above. From here we see $\mathring{H}_j = \sum_{k,\ell} \mathring{h}^{k\ell} \mathring{B}_{k\ell j} = -1$ for each j. □

Remark The previous line shows that these tori are Hamiltonian stationary but *not minimal*.

Lemma 1 suggests that we should choose a point $p \in M$, find local complex coordinates in a neighbourhood $\mathcal{V}$ of p as in Sect. 2.1, and consider the re-scaled coordinate map φ_ρ as above. We now let $r \in \mathbb{R}_+^2$ be a *unit* vector, and consider the submanifold $\Sigma_r \subset (\rho^{-1}\mathcal{V}_0, g_\rho, \omega_\rho)$, which corresponds to the "small" torus $\Sigma_{\rho r} \subset \mathcal{V}_0$, which we identify with a torus in M. The submanifold Σ_r is Hamiltonian stationary Lagrangian with respect to the standard Kähler structure, but it is no longer necessarily so with respect to the Kähler structure (g_ρ, ω_ρ, J) with Kähler potential F_ρ. However, if ρ is sufficiently small, then Σ_r is totally real; moreover, it is close, in a sense that will be made more precise later on, to being Hamiltonian stationary Lagrangian. We will perturb it to a surface which is Hamiltonian stationary Lagrangian with respect to (g_ρ, ω_ρ) for ρ small enough, and thus corresponds to a Hamiltonian stationary Lagrangian torus in (M, g, ω).

3.3 The equations to solve

An exactly Hamiltonian stationary Lagrangian submanifold with respect to the Kähler structure (g_ρ, ω_ρ, J) near the submanifold Σ_r when ρ is sufficiently small will be found by perturbing Σ_r appropriately. This will be done by first defining a class of deformations of Σ_r and then selecting the appropriate deformation by solving a differential equation. We define these deformations as follows. Let $E_k := ir_k e^{i\theta^k} \frac{\partial}{\partial z^k}$ be the coordinate basis vectors of the tangent space $T\Sigma_r$. For every function $(X^1, X^2) : \mathbb{T}^2 \to \mathbb{R}^2$ of suitably small norm, define an embedding $\mu_X : \Sigma_r \cong \mathbb{T}^2 \hookrightarrow \mathbb{C}^2$ by

$$\mu_X : (\theta^1, \theta^2) \longmapsto \left(r_1(1 - X^1(\theta))e^{i\theta^1}, r_2(1 - X^2(\theta))e^{i\theta^2} \right)$$

where we let $X = X^1 J E_1 + X^2 J E_2 \in \Gamma(J(T\Sigma_r))$. Note that the Euclidean-normal bundle of Σ_r coincides with the bundle $J(T\Sigma_r)$ and is spanned by the Euclidean-orthonormal vector fields $N_k := e^{i\theta^k} \frac{\partial}{\partial z^k}$ for $k = 1, 2$, and we can write $X = -r_1 X^1 N_1 - r_2 X^2 N_2$. Thus a geometric interpretation of this embedding is to view X (built from (X^1, X^2) by scaling by the radii r_1, r_2 in the different coordinate directions) as the section of the bundle $J(T\Sigma_r)$, so that μ_X is the Euclidean-exponential map.

Finding $X \in \Gamma(J(T\Sigma_r))$ so that $\mu_X(\Sigma_r)$ is Hamiltonian stationary Lagrangian with respect to the Kähler structure (g_ρ, ω_ρ, J) amounts to solving two equations:

$$\begin{aligned} \mu_X^* \omega_\rho &= 0 \\ \nabla \cdot H(\mu_X(\Sigma_r)) &= 0 \end{aligned} \tag{8}$$

where $H(\mu_X(\Sigma_r))$ is the symplectic mean curvature one-form. Thus one should consider the differential operator $\Phi_\rho : \Gamma(J(T\Sigma_r)) \to \Lambda^2(\Sigma_r) \times \Lambda^0(\Sigma_r)$, defined on an open neighborhood of zero, given by

$$\Phi_\rho(X) := \mu_X^* \left(\omega_\rho, \nabla \cdot H(\mu_X(\Sigma_r)) \right)$$

and attempt to solve the equation $\Phi_\rho(X) = (0, 0)$. Note that the first of these equations is first-order in the vector field X while the second equation is third-order in X. Since Σ_r is generally neither Hamiltonian stationary nor Lagrangian with respect to the Kähler structure (g_ρ, ω_ρ, J) when $\rho > 0$, then $\Phi_\rho(0)$ is a tensor field on Σ_r depending continuously on ρ in

some way that will be determined in the sequel. Certainly, however, since (g_ρ, ω_ρ) converges to the standard structure as $\rho \to 0^+$, one can assert that $\Phi_0(0) := \lim_{\rho \to 0^+} \Phi_\rho(0) = (0, 0)$.

It turns out that, as it stands, Eq. 8 does not represent a strictly elliptic problem. A few refinements are necessary in order to achieve this. First, an important observation to make is that the operator Φ_ρ maps onto a much smaller space. In fact, it is true that the first component of $\Phi_\rho(X)$ belongs to $d\Lambda^1(\Sigma_r)$, the set of exact one-forms, which can be seen as follows. Observe that $\mu_X^* \omega_\rho$ is closed and belongs to the same cohomology class as $\mu_{tX}^* \omega_\rho$ for all $t \in [0, 1]$. In fact this is a family of exact forms, which can be seen by pulling back the local Liouville form, e.g. $\mu_0^* \omega_\rho = d\alpha_\rho \big|_{\Sigma_r}$ where α_ρ is the Liouville form of ω_ρ in this coordinate chart. The second factor of $\Phi_\rho(X)$ is a divergence; hence its integral against the volume form of $\mu_X(\Sigma_r)$ must vanish.

Next, we make an *Ansatz* for the section X of the bundle $J(T\Sigma_r)$. We write $X := X^1 J E_1 + X^2 J E_2$ where again $E_k := i r_k e^{i\theta^k} \frac{\partial}{\partial z^k}$ are the coordinate basis vectors of the tangent space $T\Sigma_r$, and motivated by the Hodge decomposition, we split X into a gradient and a curl component with respect to the metric induced on Σ_r by the Euclidean ambient metric. In particular, we restrict to X with vanishing harmonic component; the harmonic vector fields correspond to non-Hamiltonian deformations of the torus (cf. the discussion at the end of Sect. 4.2). More specifically, we choose $X := \mathcal{X}(u, v)$ so that $X \lrcorner \mathring{\omega} \big|_{\Sigma_r} = dv + \mathring{\star} du$ for functions $u, v : \Sigma_r \to \mathbb{R}$, where $\mathring{\star}$ is the Hodge star operator of Σ_r with respect to the Euclidean metric. By inspection, this outcome is achieved by the vector field

$$\mathcal{X}(u, v) := \sum_{k=1}^{2} \frac{1}{r_k} \left(\frac{\partial v}{\partial \theta^k} + \sum_j \varepsilon_k^j \frac{\partial u}{\partial \theta^j} \right) e^{i\theta^k} \frac{\partial}{\partial z^k} \tag{9}$$

where ε_k^j satisfies $\varepsilon_1^1 = \varepsilon_2^2 = 0$ and $\varepsilon_1^2 = -r_1/r_2$ and $\varepsilon_2^1 = r_2/r_1$. Note that the mapping given by $(u, v) \mapsto \mathcal{X}(u, v)$ is linear in (u, v) and independent of ρ (recall $\|r\| = 1$ after re-scaling).

Using the *Ansatz* above, one can re-formulate (8) as a pair of equations for the functions u and v which will turn out to be elliptic. Since (8) is a mixed first- and third-order partial differential equation and $\mathcal{X}(u, v)$ takes one additional derivative, the functions u and v will be assumed to lie in $C^{4,\alpha}$. Moreover, since $\mathcal{X}(u, v)$ clearly remains unchanged if a constant is added to either u or v, we impose the normalization

$$\int_{\Sigma_r} u \, d\text{Vol}_{\Sigma_r}^\circ = \int_{\Sigma_r} v \, d\text{Vol}_{\Sigma_r}^\circ = 0 \tag{10}$$

where $d\text{Vol}_{\Sigma_r}^\circ$ is the volume form of Σ_r with respect to the metric induced on Σ_r by the ambient Euclidean metric. Therefore define a new differential operator, defined on an open neighborhood of $(0, 0)$, by

$$\Phi_\rho : C_0^{4,\alpha}(\Sigma_r) \times C_0^{4,\alpha}(\Sigma_r) \to C^{2,\alpha}(d\Lambda^1(\Sigma_r)) \times C^{0,\alpha}(\Sigma_r)$$
$$\Phi_\rho(u, v) := \Phi_\rho \circ \mathcal{X}(u, v)$$

where we use the zero subscript to denote a function space upon which our normalization (10) has been imposed.

4 Analysis of the Hamiltonian stationary Lagrangian operator

In order to solve the equation $\mathbf{\Phi}_\rho(u, v) = (0, 0)$ perturbatively, it is necessary to understand the mapping properties of the linearization $D_{(0,0)}\mathbf{\Phi}_\rho$ of the operator $\mathbf{\Phi}_\rho$ at $(0, 0)$. We will use the notation $\boldsymbol{L}_\rho := D_{(0,0)}\mathbf{\Phi}_\rho$ as well as $L_\rho := D_0\Phi_\rho$ in the remainder of the paper. Observe that $\boldsymbol{L}_\rho = L_\rho \circ \mathcal{X}$ by linearity. Furthermore, since $\mathbf{\Phi}_\rho$ for $\rho > 0$ will often be compared with its Euclidean analogue at $\rho = 0$, we introduce the notation $\mathring{\mathbf{\Phi}} := \mathbf{\Phi}_0$ and $\mathring{\Phi} := \Phi_0$ in keeping with the convention of adorning objects associated with the Euclidean metric with "∘". Thus we shall denote the linearizations of these operators by $\mathring{L} := D_0\mathring{\Phi}$ and $\mathring{\boldsymbol{L}} := \mathring{L} \circ \mathcal{X}$, respectively.

This section contains the following material. First we compute linearized operator $\mathring{\boldsymbol{L}}$ and determine its kernel. It will turn out that $\mathring{\boldsymbol{L}}$ is not self-adjoint; hence we next compute the adjoint $\mathring{\boldsymbol{L}}^*$ and compute its kernel. Finally, we compute $\boldsymbol{L}_\rho$ with enough detail to be able to give estimates, in terms of ρ, for the difference $\boldsymbol{P}_\rho := \boldsymbol{L}_\rho - \mathring{\boldsymbol{L}}$.

4.1 The unperturbed linearization

Let $\mathring{\mathbf{\Phi}}$ be the Hamiltonian stationary Lagrangian differential operator with respect to the standard Kähler structure $(\mathring{g}, \mathring{\omega}, J)$. The task at hand is to compute its linearization at zero, denoted by $\mathring{\boldsymbol{L}}$. Since $\mathring{\mathbf{\Phi}} = \mathring{\Phi} \circ \mathcal{X}$ and $\mathcal{X}$ is linear, the main computation is to find the linearization at zero of $\mathring{\Phi}$ acting on sections X of $J(T\Sigma_r)$, denoted by $\mathring{L}$ and given by

$$\mathring{L}(X) := \frac{d}{dt}\mu_t^*\left(\mathring{\omega}\big|_{\Sigma_r}, \mathring{\nabla}\cdot\mathring{H}(\mu_t(\Sigma_r))\right)\bigg|_{t=0} := \left(\mathring{L}^{(1)}(X), \mathring{L}^{(2)}(X)\right)$$

where $\mu_t : \mathbb{C}^2 \to \mathbb{C}^2$ is a family of diffeomorphisms generating X. In the computations below, repeated upper and lower indices are summed, a comma denotes ordinary differentiation and a semi-colon denotes covariant differentiation with respect to the induced metric.

Proposition 2 *Let $\Sigma \subset \mathbb{C}^n$ be Lagrangian for the standard symplectic structure. Let X be a C^3 section of $N(\Sigma) = J(T\Sigma)$, generated by a family of diffeomorphisms $\mu_t : \mathbb{C}^n \to \mathbb{C}^n$, and let $\mathring{L}(X) := \frac{d}{dt}\mu_t^*\left(\mathring{\omega}\big|_\Sigma, \mathring{\nabla}\cdot\mathring{H}(\mu_t(\Sigma))\right)\bigg|_{t=0} := \left(\mathring{L}^{(1)}(X), \mathring{L}^{(2)}(X)\right)$. Write $X := X^j J E_j$ where $\{E_1, E_2, \ldots, E_n\}$ is a coordinate basis for the tangent space of Σ. Then the following hold (where the quantities are computed in the induced metric on Σ):*

$\mathring{L}^{(1)}(X) = \mathrm{d}\left(X \lrcorner \mathring{\omega}\right)$

$\mathring{L}^{(2)}(X) = -(\mathring{\Delta}X^m)_{;m} - \mathring{h}^{\ell s}\mathring{h}^{mq}X^u\mathring{B}_{squ}\mathring{H}_{\ell;m} - \mathring{h}^{\ell m}\mathring{h}^{sk}\mathring{H}_s\left(X^u\mathring{B}_{\ell ku}\right)_{;m} + \mathring{h}^{km}\mathring{H}_k\left(X^u\mathring{H}_u\right)_{;m}$

$\qquad\qquad - \mathring{h}^{\ell m}\mathring{h}^{js}\mathring{h}^{kq}\left(X^u\mathring{B}_{squ}\mathring{B}_{jk\ell}\right)_{;m}.$

Proof The formula for $\mathring{L}^{(1)}$ is straightforward. Recall that it is a standard computation involving the Lie derivative of a 2-form to show that $\frac{d}{dt}\mu_t^*\mathring{\omega}\big|_{t=0} = \mathrm{d}(X \lrcorner \mathring{\omega}) + X \lrcorner \mathrm{d}\mathring{\omega}$. Since $\mathrm{d}\mathring{\omega} = 0$, then $\mathring{L}^{(1)}(X) = \mathrm{d}\left(X \lrcorner \mathring{\omega}\right)$ follows as desired.

The remainder of the proof is the computation of $\mathring{L}^{(2)}(X)$. Let Σ be a Lagrangian submanifold of $\mathbb{C}^n$ carrying the Euclidean metric $\mathring{g}$, and let X be a section of the normal bundle of Σ. Extend X off of Σ, let $\mu_t : \mathbb{C}^n \to \mathbb{C}^n$ be a one-parameter family of diffeomorphisms with $\frac{d}{dt}\mu_t\big|_{t=0} = X$ and set $\Sigma^t := \mu_t(\Sigma)$. Next, choose $\{E_1, E_2, \ldots, E_n\}$ a local coordinate frame for Σ coming from geodesic normal coordinates at $p_0 \in \Sigma$ in the induced metric $\mathring{h}$ at

$t = 0$. Then $\{JE_1, JE_2, \ldots, JE_n\}$ is a basis for the normal bundle of Σ at $t = 0$, because Σ is Lagrangian. But this does not necessarily hold for $|t| \neq 0$, since μ_t is not assumed to be a family of symplectomorphisms. However, for p near p_0, $T_p\mathbb{C}^n = T_p\Sigma \oplus J(T_p\Sigma)$. We write $X = X^j JE_j$ along Σ, and $\mathring{\nabla}_{E_\ell}(X^j E_j) = X^j_{;\ell} E_j$, and we may assume $X(p_0) \neq 0$. Note that X and E_k commute along μ_t, and since X is transverse to Σ, we can extend the fields E_k locally using the diffeomorphism μ_t to a basis for $T_{\mu_t(p)}\Sigma^t$, for $|t|$ small. In these coordinates the matrix for $\mathring{h}$ on Σ^t is the same as that for $\mu_t^* \mathring{g}$ on $T\Sigma$. The computations below are evaluated at p_0 at $t = 0$.

In terms of the local coordinates introduced above, we have

$$\mathring{\nabla} \cdot \mathring{H}(\Sigma^t) = \mathring{h}^{\ell m} \mathring{h}^{jk} \mathring{B}_{jk\ell;m} \tag{11}$$

where $\mathring{h}_{k\ell} := \mathring{g}(E_k, E_\ell)$ is the induced metric, $\mathring{h}^{jk}$ are the components of the inverse of the induced metric, $\mathring{\nabla}$ is the induced connection, and

$$\mathring{B}_{jk\ell} = \mathring{\omega}((\mathring{D}_{E_j} E_k)^\perp, E_\ell) = \mathring{\omega}(\mathring{D}_{E_j} E_k, E_\ell) - \mathring{\Gamma}^s_{jk}\mathring{\omega}(E_s, E_\ell)$$

with $\mathring{\Gamma}^s_{jk}$ the Christoffel symbols of $\mathring{h}_{jk}$ and $\mathring{D}$ the ambient Euclidean connection.

The terms in (11) all depend on t. Since $\mathring{\nabla} \cdot \mathring{H}(\Sigma^t) = \mathring{h}^{\ell m} \mathring{H}_{\ell;m} = \mathring{h}^{\ell m} \mathring{H}_{\ell,m} - \mathring{h}^{\ell m} \mathring{\Gamma}^s_{\ell m} \mathring{H}_s$ where $\mathring{H}_\ell := \mathring{h}^{jk} \mathring{B}_{jk\ell}$, differentiating (11) at $t = 0$ yields

$$\frac{d}{dt}(\mathring{\nabla} \cdot \mathring{H}(\Sigma^t))\Big|_{t=0} = (\mathring{h}^{\ell m})' \mathring{H}_{\ell;m} - \mathring{h}^{\ell m}(\mathring{\Gamma}^s_{\ell m})' \mathring{H}_s + \mathring{h}^{\ell m}\left((\mathring{H}_\ell)'\right)_{;m}$$

where a prime denotes the value of the t-derivative at $t = 0$.

Expressions for $(\mathring{h}^{\ell m})'$ and $(\mathring{\Gamma}^s_{\ell m})'$ and $(\mathring{H}_\ell)'$ are now required. To begin, it is straightforward to compute

$$(\mathring{h}^{\ell m})' = -2\mathring{h}^{\ell s}\mathring{h}^{mq} X^u \mathring{B}_{squ}$$

$$(\mathring{\Gamma}^s_{\ell m})' = \mathring{h}^{sq}\left(\left(X^u \mathring{B}_{\ell qu}\right)_{;m} + \left(X^u \mathring{B}_{mqu}\right)_{;\ell} - \left(X^u \mathring{B}_{\ell mu}\right)_{;q}\right).$$

Next

$$\left(\mathring{H}_\ell\right)' = (\mathring{h}^{jk})' \mathring{B}_{jk\ell} + \mathring{h}^{jk}(\mathring{B}_{jk\ell})' = -2\mathring{h}^{js}\mathring{h}^{kq} X^u \mathring{B}_{squ}\mathring{B}_{jk\ell} + \mathring{h}^{jk}(\mathring{B}_{jk\ell})'$$

and the fact that both $\mathring{\Gamma}^s_{jk}(p_0)$ and $\mathring{\omega}\big|_{\Sigma^t}$ vanish at $t = 0$ implies

$$(\mathring{B}_{jk\ell})' = \frac{d}{dt}\left(\mathring{\omega}\left(\mathring{D}_{E_j} E_k, E_\ell\right) - \mathring{\Gamma}^s_{jk}\mathring{\omega}(E_s, E_\ell)\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(\mathring{\omega}\left(\mathring{D}_{E_j} E_k, E_\ell\right)\right)\Big|_{t=0}$$

$$= \mathring{\omega}\left(\mathring{D}_X\mathring{D}_{E_j} E_k, E_\ell\right) + \mathring{\omega}\left(\mathring{D}_{E_j} E_k, \mathring{D}_X E_\ell\right)$$

$$= \mathring{\omega}\left(\mathring{D}_{E_j}\mathring{D}_{E_k} X, E_\ell\right) + \mathring{\omega}\left(\mathring{D}_{E_j} E_k, \mathring{D}_{E_\ell} X\right)$$

$$= E_j\mathring{\omega}\left(\mathring{D}_{E_k} X, E_\ell\right) - \mathring{\omega}\left(\mathring{D}_{E_k} X, \mathring{D}_{E_j} E_\ell\right) + \mathring{\omega}\left(\mathring{D}_{E_j} E_k, \mathring{D}_{E_\ell} X\right)$$

$$= -E_j\mathring{g}\left(\mathring{D}_{E_k}(X^q E_q), E_\ell\right) + \mathring{g}\left(\mathring{D}_{E_k}(X^q E_q), \mathring{D}_{E_j} E_\ell\right) + \mathring{g}\left(\mathring{D}_{E_j} E_k, \mathring{D}_{E_\ell}(X^q E_q)\right).$$

We have used that X commutes with E_k along μ_t, that the ambient curvature vanishes, and that $\mathring{\omega}$ and J are parallel. Now

$$\mathring{D}_{E_\ell}(X^q E_q) = X^q_{,\ell} E_q + X^q \mathring{D}_{E_\ell} E_q = X^q_{;\ell} E_q - X^q \mathring{h}^{uv} \mathring{B}_{\ell qu} J E_v.$$

Note that at $t = 0$, $\mathring{D}_{E_k} E_j$ is normal to Σ at p_0, and moreover $\mathring{g}(\mathring{D}_{E_j} E_k, J E_m) = -\mathring{B}_{jkm}$ at p_0. Thus we have

$$(\mathring{B}_{jk\ell})' = -E_j \mathring{g}\left(X^q_{;k} E_q - X^q \mathring{h}^{uv} \mathring{B}_{kqu} J E_v, E_\ell\right) + \mathring{g}\left(X^q_{;k} E_q - X^q \mathring{h}^{uv} \mathring{B}_{kqu} J E_v, \mathring{D}_{E_j} E_\ell\right)$$

$$+ \mathring{g}\left(\mathring{D}_{E_j} E_k, X^q_{;\ell} E_q - X^q \mathring{h}^{uv} \mathring{B}_{\ell qu} J E_v\right)$$

$$= -X^q_{;kj} \mathring{h}_{q\ell} + X^q \mathring{B}_{kqu} \mathring{B}_{j\ell v} \mathring{h}^{uv} + X^q \mathring{B}_{jkv} \mathring{B}_{\ell qu} \mathring{h}^{uv}.$$

Everything can now be put together:

$$\mathring{L}^{(2)}(X) = -2\mathring{h}^{\ell s} \mathring{h}^{mq} X^u \mathring{B}_{squ} \mathring{H}_{\ell;m}$$

$$-\mathring{h}^{sq} \mathring{H}_s \left(2\left(X^u \mathring{B}_{\ell qu}\right)_{;m} \mathring{h}^{\ell m} - \left(X^u \mathring{H}_u\right)_{;q}\right)$$

$$-2\mathring{h}^{\ell m} \mathring{h}^{js} \mathring{h}^{kq} \left(X^u \mathring{B}_{squ} \mathring{B}_{jk\ell}\right)_{;m}$$

$$+\mathring{h}^{\ell m} \mathring{h}^{jk} \left(-X^q_{;kj} \mathring{h}_{q\ell} + X^q \mathring{B}_{kqu} \mathring{B}_{j\ell v} \mathring{h}^{uv} + X^q \mathring{B}_{jkv} \mathring{B}_{\ell qu} \mathring{h}^{uv}\right)_{;m}$$

$$= -(\mathring{\Delta} X^m)_{;m} - \mathring{h}^{\ell s} \mathring{h}^{mq} X^u \mathring{B}_{squ} \mathring{H}_{\ell;m} - \mathring{h}^{sq} \mathring{H}_s \left(\left(X^u \mathring{B}_{\ell qu}\right)_{;m} \mathring{h}^{\ell m} - \left(X^u \mathring{H}_u\right)_{;q}\right)$$

$$-\mathring{h}^{\ell m} \mathring{h}^{js} \mathring{h}^{kq} \left(X^u \mathring{B}_{squ} \mathring{B}_{jk\ell}\right)_{;m}$$

This is the desired formula. $\qquad\square$

To compute $\mathring{L}^{(2)}$ for the torus Σ_r, note that both $\mathring{B}$ and $\mathring{H}$ are parallel tensors in this case. Consequently the second fundamental form terms in $\mathring{L}^{(2)}$ become simply $X \mapsto -\mathring{A}^m_\ell X^\ell_{;m}$ where

$$\mathring{A}^{\ell m} := \mathring{H}_s \mathring{B}^{\ell sm} - \mathring{H}^\ell \mathring{H}^m + \mathring{B}^\ell_{sq} \mathring{B}^{sqm}$$

and furthermore, we can compute this precisely: substituting $\mathring{h}_{k\ell} = r_k^2 \delta_{k\ell}$ and $\mathring{B}_{jk\ell} = -r_s^2 \delta_{sj} \delta_{sk} \delta_{s\ell}$ for the induced metric and symplectic second fundamental form of Σ_r (using θ-coordinates) with respect to the Euclidean metric yields

$$\mathring{A}^{\ell m} = \frac{2\delta_{\ell m}}{r_\ell^2 r_m^2} - \frac{1}{r_\ell^2 r_m^2}.$$

We now let $X = \mathcal{X}(u, v)$ as in (9) and substitute this into the formulæ of Proposition 2 to find the linearization $\mathring{L}$.

Corollary 3 *Let $(u, v) \in C_0^{4,\alpha}(\Sigma_r) \times C_0^{4,\alpha}(\Sigma_r)$. Write $\mathring{L} = \left(\mathring{L}^{(1)}, \mathring{L}^{(2)}\right)$. Then*

$$\mathring{L}^{(1)}(u, v) := \mathrm{d} \mathring{\star} \, \mathrm{d}u$$

$$\mathring{L}^{(2)}(u, v) := \mathring{\Delta}(\mathring{\Delta}v) + \mathring{A}^{\ell m} v_{;\ell m} + \mathring{A}^{\ell m} \varepsilon^k_\ell u_{;mk}.$$

4.2 The kernel of the unperturbed linearization

The determination of the kernel of the linearized operator $\mathring{L}$ is best done in two stages. First one finds the kernel of $\mathring{L}$ and then one takes into account the effect of $\mathcal{X}$. Thus the starting point is to express the formulæ of Proposition 2 explicitly in local coordinates. To this end, suppose that Σ_r is given in local coordinates by its standard embedding. Make the *Ansatz* $X := \sum_k X^k(-r_k e^{i\theta^k} \frac{\partial}{\partial z^k})$ for the deformation vector field in the formulæ from Proposition 2 to obtain

$$\mathring{L}(X) = -\left(\sum_{j,k} r_k^2 X^k_{,j} d\theta^j \wedge d\theta^k, \quad \sum_{j,k} \frac{1}{r_k^2} \left(X^j_{,kkj} - X^j_{,k} \right) + \sum_j \frac{2}{r_j^2} X^j_{,j} \right).$$

The operator $\mathring{L}$ thus becomes a constant-coefficient differential operator on the torus. Solving the equation $\mathring{L}(X) = (0,0)$ for the kernel of $\mathring{L}$ thus becomes a matter of Fourier analysis. The following proposition appears in [11] for the n-dimensional torus; the $n = 2$ case is included here for the sake of completeness.

Proposition 4 *Expressed in the local coordinates for the standard embedding of Σ_r, the kernel of $\mathring{L}$ consists of vector fields $X := \sum_k X^k(-r_k e^{i\theta^k} \frac{\partial}{\partial z^k})$ where*

$$X^k = \lambda_k + \frac{1}{r_k^2} \frac{\partial f}{\partial \theta^k}$$

with $f(\theta) := a + \sum_j \left(b_{j1} \cos(\theta^j) + b_{j2} \sin(\theta^j) \right) + c_1 \cos(\theta^1 - \theta^2) + c_2 \sin(\theta^1 - \theta^2)$, for $a, b_{js}, c_s, \lambda_k \in \mathbb{R}$.

Proof The first equation in $\mathring{L}(X) = (0,0)$ implies either of the following (or a linear combination thereof): that X^k is constant for every k, and thus the one-form $r_k^2 X^k d\theta^k$ is harmonic on Σ_r; or else that there is a function $f : \mathbb{T}^2 \to \mathbb{R}$ with

$$X^k = \frac{1}{r_k^2} \frac{\partial f}{\partial \theta^k}.$$

In the first case, the second equation in $\mathring{L}(X) = (0,0)$ is satisfied trivially. Note that a one-form of this type is *not* exact, implying that X is *not* induced by a Hamiltonian vector field. In the second case, insert $X^k := r_k^{-2} \frac{\partial f}{\partial \theta^k}$ into the second equation to find

$$\sum_{j,k} \frac{1}{r_j^2 r_k^2} \left(f_{,jjkk} - f_{,jk} \right) + \sum_j \frac{2}{r_j^4} f_{,jj} = 0.$$

This is a constant-coefficient, fourth order elliptic equation on the torus which can be solved by taking the discrete Fourier transform. The Fourier coefficients $\hat{f}(\vec{n}) := \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(\theta) e^{-i\vec{\theta}\cdot\vec{n}} d^2\theta$ of the solutions must thus satisfy

$$\left(\sum_{j,k} \frac{n_j^2 n_k^2 + n_j n_k}{r_j^2 r_k^2} - \sum_j \frac{2n_j^2}{r_j^4} \right) \hat{f}(\vec{n}) = 0.$$

Thus $\hat{f}(\vec{n}) = 0$ unless $\vec{n} = (n_1, n_2)$ solves the equation obtained by setting the coefficient of $\hat{f}(\vec{n})$ to zero. One solution of this equation is $n_1 = n_2 = 0$, and this corresponds to the constant functions. There are also non-trivial solutions of this equation: either $n_j = \pm 1$ for

some fixed j and all other $n_k = 0$; or else $n_j = \pm 1$ and $n_k = \mp 1$ for $j \neq k$. The fact that there are no other non-trivial solutions can be seen as follows. Summing over $j, k \in \{1, 2\}$ explicitly and re-arranging terms yields the equation $n_1^2 \pm n_1 + r_1^2 r_2^{-2} \left(n_2^2 + n_2 \right) = 0$. But since the quadratic $x^2 \pm x + C^2$ only has the integer roots $x = 0, 1$ when $C = 0$ and no integer roots when $C \neq 0$, it must be the case that $(n_1, n_2) = (1, 0), (0, 1), (1, -1)$ or $(-1, 1)$. Applying the inverse Fourier transform now yields the desired vector fields in the kernel of $\mathring{L}$. □

Observe that there is a geometric interpretation of the kernel of $\mathring{L}$. The one-parameter families of complex structure-preserving isometries of $\mathbb{C}^2$ are the unitary rotations and the translations. Each of these is a Hamiltonian deformation where the Hamiltonians are given by linear functions in the first case and quadratic polynomials of the form $z \mapsto z^* \cdot A \cdot z$ in the second case, where A is a Hermitian matrix. Of these, only the non-diagonal Hermitian matrices generate non-trivial motions of Σ_r. In fact, let (U, τ) denote the motion of $\mathbb{C}^2$ given by $z \mapsto U(z) + \tau$ where $U \in U(2)$ and $\tau \in \mathbb{C}^2$. Then we consider the six-dimensional parameter family of motions of $\mathbb{C}^2$ given by

$$\mathcal{R} := \left\{ (\exp(\tau_5 K_1 + \tau_6 K_2), \tau) : \tau_5, \tau_6 \in \mathbb{R} \quad \text{and} \quad \tau := (\tau_1, \dots, \tau_4) \in \mathbb{R}^4 \right\}$$

where

$$K_1 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad K_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

are elements in the Lie algebra of $U(2)$ that generate all non-trivial $U(2)$-rotations of Σ_r. Denote by $\mathring{\mu}_t^{(i)}$ for $i = 1, \dots, 6$, those motions which correspond to $\tau_i = t$ and $\tau_{i'} = 0$ for $i' \neq i$. Note that each $\mathring{\mu}^{(i)}$ is Hamiltonian with respect to the Euclidean Kähler structure, with $J\mathring{\nabla} f^{(j)} := \frac{d}{dt} \mathring{\mu}_t^{(j)} \big|_{t=0}$, where we let $\mathring{\nabla} f$ be the ambient gradient vector of f. Indeed, the translations $\mathring{\mu}_t^{(j)}$, $1 \leq j \leq 4$, yield Hamiltonian functions which restricted to Σ_r are just (up to sign) $r_s \cos(\theta^s)$ and $r_s \sin(\theta^s)$ for $s = 1, 2$, while the $U(2)$-rotations $\mathring{\mu}_t^{(5)}$ and $\mathring{\mu}_t^{(6)}$ yield Hamiltonian functions which restrict to Σ_r as (again, up to sign) $r_1 r_2 \sin(\theta^1 - \theta^2)$ and $r_1 r_2 \cos(\theta^1 - \theta^2)$. The span of the restrictions of these Hamiltonian functions to Σ_r are the functions in the kernel of $\mathring{L}$ of the form

$$f(\theta) = \sum_j \left(b_{j1} r_j \cos(\theta^j) + b_{j2} r_j \sin(\theta^j) \right) + c_1 r_1 r_2 \cos(\theta^1 - \theta^2) + c_2 r_1 r_2 \sin(\theta^1 - \theta^2)$$

(12)

for $b_{js}, c_s \in \mathbb{R}$. The remaining elements of the kernel of $\mathring{L}$ derive from another set of deformations of Σ_r which preserve both the Lagrangian condition and the Hamiltonian-stationarity. These arise from allowing the radii of Σ_r to vary—in other words the deformations $\Sigma^t := \Sigma_{r+at}$ for some $a = (a_1, a_2)$.

Corollary 5 *The kernel of $\mathring{L}$ is*

$$\mathcal{K} := \{0\} \times \text{span}_{\mathbb{R}} \{ r_1 \cos(\theta^1), r_1 \sin(\theta^1), r_2 \cos(\theta^2),$$
$$r_2 \sin(\theta^2), r_1 r_2 \cos(\theta^1 - \theta^2), r_1 r_2 \sin(\theta^1 - \theta^2) \}.$$

Note: The constant functions are not in $\mathcal{K}$ because the conditions $\int_{\Sigma_r} u \, d\text{Vol}^\circ_{\Sigma_r} = \int_{\Sigma_r} v \, d\text{Vol}^\circ_{\Sigma_r} = 0$ have been imposed on functions in the domain of $\mathring{L}$.

4.3 The adjoint of the unperturbed linearization

The operator $\mathring{L}$ computed in Sect. 4.1 is not self-adjoint. Thus it is necessary to compute the adjoint and find its kernel in order to determine a space onto which $\mathring{L}$ is surjective.

Proposition 6 *The formal L^2 adjoint of $\mathring{L}$: $C_0^{4,\alpha}(\Sigma_r) \times C_0^{4,\alpha}(\Sigma_r) \to C^{2,\alpha}(\mathrm{d}\Lambda^1(\Sigma_r)) \times C^{0,\alpha}(\Sigma_r)$ is the operator $\mathring{L}^* := \left([\mathring{L}^*]^{(1)}, [\mathring{L}^*]^{(2)}\right)$: $C^{4,\alpha}(\mathrm{d}\Lambda^1(\Sigma_r)) \times C^{4,\alpha}(\Sigma_r) \to C_0^{2,\alpha}(\Sigma_r) \times C_0^{0,\alpha}(\Sigma_r)$ where*

$$
\begin{aligned}
[\mathring{L}^*]^{(1)}(\mathring{\star}u, v) &:= \mathring{\Delta}u + \mathring{A}^{\ell m}\varepsilon_\ell^k v_{;mk} \\
[\mathring{L}^*]^{(2)}(\mathring{\star}u, v) &:= \mathring{\Delta}(\mathring{\Delta}v) + \mathring{A}^{\ell m}v_{\ell m}.
\end{aligned}
\tag{13}
$$

and $\mathring{A}^{\ell m} = 2r_\ell^{-2}r_m^{-2}\delta^{\ell m} - r_\ell^{-2}r_m^{-2}$ as computed earlier.

Proof Straightforward integration by parts based on the formulæ for $\mathring{L}$ and $\mathcal{X}$. □

The kernel $\mathcal{K}^*$ of the adjoint $\mathring{L}^*$ is now easy to find, given the formula (13). Consider the equation $\mathring{L}^*(\mathring{\star}u, v) = (0, 0)$ for $(u, v) \in C_0^{4,\alpha}(\Sigma_r) \times C^{4,\alpha}(\Sigma_r)$. The second of these equations along with the calculations of Sect. 4.2 implies that v is of the form (12) found before. Now u can be determined from the first of these equations via $\mathring{\Delta}u = -\mathring{A}^{\ell m}\varepsilon_\ell^k v_{;mk}$. Since the form of $\mathring{A}^{\ell m}$ is known, one can in fact determine u explicitly. Note that we will employ a slight abuse of notation below by identifying $C_0^{k,\alpha}(\Sigma_r)$ with $C^{k,\alpha}(\mathrm{d}\Lambda^1(\Sigma_r))$ via the Hodge star operator. Furthermore, we incorporate certain constant factors into the basis elements for reasons that will become clear later.

Corollary 7 *The kernel of $\mathring{L}^*$ is*

$$
\mathcal{K}^* := \mathrm{span}_\mathbb{R}\{(0, 1)\} \oplus \mathrm{span}_\mathbb{R}\Big\{r_1\cos(\theta^1)\cdot w^{(1)}, r_1\sin(\theta^1)\cdot w^{(2)},
$$

$$
r_2\cos(\theta^2)\cdot w^{(3)}, r_2\sin(\theta^2)\cdot w^{(4)}, r_1r_2\cos(\theta^1 - \theta^2)\cdot w^{(5)}, r_1r_2\sin(\theta^1 - \theta^2)\cdot w^{(6)}\Big\}
$$

where $w^{(1)} = w^{(2)} = \left((r_1r_2)^{-1}, 1\right)$ and $w^{(3)} = w^{(4)} = \left(-(r_1r_2)^{-1}, 1\right)$ and $w^{(5)} = w^{(6)} = (0, \rho)$.

4.4 The perturbed linearization

Let $\mathbf{\Phi}_\rho$ be the Hamiltonian stationary Lagrangian differential operator with respect to the Kähler structure (g_ρ, ω_ρ, J) corresponding to the Kähler potential $F_\rho(z, \bar{z}) = \frac{1}{2}\|z\|^2 + \rho^2\hat{F}_\rho(z, \bar{z})$ with $\rho > 0$. The present goal is to compute its linearization $\mathbf{L}_\rho$ at zero, and express it as a perturbation of $\mathring{L}$ in the form $\mathbf{L}_\rho = \mathring{L} + \mathbf{P}_\rho$. Then the dependence of $\mathbf{P}_\rho$ on ρ must be analyzed. Since $\mathbf{\Phi}_\rho = \Phi_\rho \circ \mathcal{X}$ and $\mathcal{X}$ is linear, once again it is best to start with the linearization L_ρ of Φ_ρ acting on sections X of $J(T\Sigma_r)$.

We again employ the conventions that repeated indices are summed, a comma denotes ordinary differentiation and a semi-colon denotes covariant differentiation with respect to the induced metric.

Proposition 8 *Let Σ be a totally real submanifold of a Kähler manifold (M, g, ω). Let X be a C^3 section of $J(T\Sigma)$, generated by a family of diffeomorphisms $\mu_t : M \to M$. Let*

$L(X) = \frac{d}{dt}\mu_t^*\left(\omega\,|_\Sigma, \nabla \cdot H(\mu_t(\Sigma))\right)|_{t=0}$. Write $X := X^j J E_j$ where $\{E_1, E_2, \ldots, E_n\}$ is a coordinate basis for the tangent space of Σ. Then $L(X) := \left(L^{(1)}(X), L^{(2)}(X)\right)$ is given by

$$L^{(1)}(X) := \mathrm{d}\,(X \lrcorner \,\omega)$$
$$L^{(2)}(X) := \mathcal{E}_1(X) + \mathcal{E}_2(X)$$

where the following hold (and quantities are computed in the induced metric along Σ):

$$
\begin{aligned}
\mathcal{E}_1(X) :=& -(\Delta X^m)_{;m} - h^{\ell m} X^s \mathcal{R}_{s\ell} - h^{\ell m} h^{qu} H_{q;m} X^s B_{us\ell} \\
& + h^{\ell m} h^{jk} h^{uq} \left(X^s (B_{ksq} B_{j\ell u} - B_{ksq} B_{ju\ell} - B_{qsk} B_{ju\ell})\right)_{;m} \\
& - h^{\ell m} h^{uq} H_u \left(X^s B_{qs\ell}\right)_{;m} + h^{\ell m} h^{uq} H_u \left(X^s B_{\ell sm}\right)_{;q}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}_2(X) :=& -h^{\ell u} h^{mq} (h^{jk} B_{jk\ell})_{;m}\mathcal{C}(X)_{uq} - \left(h^{\ell m} h^{ju} h^{kq}\mathcal{C}(X)_{uq} B_{jk\ell}\right)_{;m} \\
& - \tfrac{1}{2} h^{\ell m} h^{jk} h^{sq} B_{jks}\left(\mathcal{C}(X)_{q\ell;m} + \mathcal{C}(X)_{rm;\ell} - \mathcal{C}(X)_{\ell m;q}\right) \\
& + h^{\ell m} h^{jk} X^s \left(g\left(\mathcal{D}((D_{E_k} E_s)^\perp), (D_{E_j} E_\ell)^\perp\right) + g\left((D_{E_j} E_k)^\perp, \mathcal{D}((D_{E_\ell} E_s)^\perp)\right)\right)_{;m} \\
& - \frac{1}{2} h^{\ell m} h^{jk} h^{sq}\omega_{s\ell}\left(\beta(X)_{qj;k} + \beta(X)_{qk;j} - \beta(X)_{jk;q}\right) \\
& + \mathcal{C}(X)_{qj;k} + \mathcal{C}(X)_{qk;j} - \mathcal{C}(X)_{jk;q})_{;m}.
\end{aligned}
$$

Here $\mathcal{C}(X)_{k\ell} := X^s_{;k}\omega_{s\ell} + X^s_{;\ell}\omega_{sk}$ and $\beta(X)_{k\ell} := X^s (B_{ks\ell} + B_{\ell sk})$. Also $\mathcal{D} : TM \to TM$ is the operator giving the difference between the orthogonal projection of a vector $W \in T_pM$ onto $N_p\Sigma$ and its orthogonal projection onto $J(T_p\Sigma)$, and $\mathcal{R}_{s\ell} = h^{jk}\mathcal{R}_{jsk\ell}$, where $\mathcal{R}_{jsk\ell} = g(R(E_j, JE_s, E_k), JE_\ell)$ and R is the ambient curvature tensor.

Proof The formula for $L^{(1)}(X)$ follows as before; thus consider $L^{(2)}(X)$. Let Σ be a totally real submanifold of M. Let X be a section of the bundle $J(T\Sigma)$, and extend X off of Σ. Let $\mu_t : M \to M$ be a one-parameter family of diffeomorphisms with $\frac{d}{dt}\mu_t|_{t=0} = X$ and set $\Sigma^t := \mu_t(\Sigma)$. Note that although X is always transverse to Σ, it is not necessarily normal to Σ because Σ is not necessarily Lagrangian.

Next, choose $\{E_1, E_2, \ldots, E_n\}$ a local coordinate frame for Σ coming from geodesic normal coordinates at $p_0 \in \Sigma$ in the induced metric h at $t = 0$. Then $\{JE_1, JE_2, \ldots, JE_n\}$ is a basis for $J(T_p\Sigma)$ for p near p_0, and $T_pM = T_p\Sigma \oplus J(T_p\Sigma)$ for such p. We write $X = X^j J E_j$ along Σ, and $\nabla_{E_\ell}(X^j E_j) = X^j_{;\ell}E_j$, and we may assume $X(p_0) \neq 0$. Note that X and E_k commute along μ_t, and since X is transverse to Σ, we can extend the fields E_k locally using the diffeomorphism μ_t to a basis for $T_{\mu_t(p)}\Sigma^t$, for $|t|$ small. In these coordinates the matrix for h on Σ^t is the same as that for $\mu_t^* g$ on $T\Sigma$. The computations below are evaluated at p_0 at $t = 0$.

In terms of these coordinates, we have

$$\nabla \cdot H(\Sigma^t) = h^{\ell m} h^{jk} B_{jk\ell;m} \tag{14}$$

where $h_{k\ell} := g(E_k, E_\ell)$ is the induced metric, h^{jk} are the components of the inverse of the induced metric, ∇ is the induced connection, and

$$B_{jk\ell} = \omega((D_{E_j} E_k)^\perp, E_\ell) = \omega(D_{E_j} E_k, E_\ell) - \Gamma^s_{jk}\omega(E_s, E_\ell),$$

where Γ^s_{jk} are the Christoffel symbols of h_{jk}, and D is the ambient connection of g.

The terms in (14) all depend on t. We will now compute the first derivative of (14) at $t = 0$. By writing

$$\nabla \cdot H(\Sigma^t) = h^{\ell m} H_{\ell;m} = h^{\ell m} H_{\ell,m} - h^{\ell m} \Gamma^s_{\ell m} H_s,$$

we find

$$\frac{d}{dt}(\nabla \cdot H(\Sigma^t))\Big|_{t=0} = (h^{\ell m})' H_{\ell;m} - h^{\ell m} (\Gamma^s_{\ell m})' H_s + h^{\ell m} ((H_\ell)')_{;m}$$

where once again a prime denotes the value of the t-derivative at $t = 0$.

We compute the first variation of the metric h. The fact that Σ is not assumed to be Lagrangian for ω influences the outcome of the computation. We have

$$\begin{aligned}
(h_{k\ell})' &= g(D_X E_k, E_\ell) + g(E_k, D_X E_\ell) \\
&= g(D_{E_k} X, E_\ell) + g(D_{E_\ell} X, E_k) \\
&= X^s_{;k} g(J E_s, E_\ell) + X^s g(J D_{E_k} E_s, E_\ell) + X^s_{;\ell} g(J E_s, E_k) + X^s g(J D_{E_\ell} E_s, E_k) \\
&= X^s_{;k} \omega_{s\ell} + X^s_{;\ell} \omega_{sk} + X^s (B_{ks\ell} + B_{\ell sk}).
\end{aligned}$$

Define $\mathcal{C}(X)_{k\ell} := X^s_{;k} \omega_{s\ell} + X^s_{;\ell} \omega_{sk}$ and $\beta(X)_{k\ell} := X^s (B_{ks\ell} + B_{\ell sk})$. Note that if Σ were Lagrangian with respect to ω then $\mathcal{C}(X)$ would vanish identically and $\beta(X)_{k\ell}$ would equal $2 X^s B_{k\ell s}$. It is now straightforward to compute

$$(h^{k\ell})' = -h^{km} h^{\ell q} h'_{mq} = -h^{km} h^{\ell q} \left(\beta(X)_{mq} + \mathcal{C}(X)_{mq} \right)$$

$$(\Gamma^k_{\ell m})' = \frac{1}{2} h^{kq} \left(\beta(X)_{q\ell;m} + \beta(X)_{qm;\ell} - \beta(X)_{\ell m;q} + \mathcal{C}(X)_{q\ell;m} + \mathcal{C}(X)_{qm;\ell} - \mathcal{C}(X)_{\ell m;q} \right).$$

Next we have

$$\begin{aligned}
(H_\ell)' &= \frac{d}{dt} \left(h^{jk} B_{jk\ell} \right)\Big|_{t=0} \\
&= -h^{jm} h^{kq} \left(\beta(X)_{mq} + \mathcal{C}(X)_{mq} \right) B_{jk\ell} + h^{jk} (B_{jk\ell})'.
\end{aligned}$$

We now use the facts that ω and J are parallel, that X and E_k commute along μ_t, and $\Gamma^s_{jk}(p_0)$ vanishes at $t = 0$ to deduce

$$\begin{aligned}
(B_{jk\ell})' &= \frac{d}{dt} \omega \left((D_{E_j} E_k)^\perp, E_\ell \right)\Big|_{t=0} \\
&= \omega \left(D_X D_{E_j} E_k, E_\ell \right) + \omega \left(D_{E_j} E_k, D_X E_\ell \right) - (\Gamma^s_{jk})' \omega_{s\ell} \\
&= \omega \left(D_{E_j} D_{E_k} X, E_\ell \right) + \omega \left(D_{E_j} E_k, D_{E_\ell} X \right) + \omega(R(E_j, X) E_k, E_\ell) - (\Gamma^s_{jk})' \omega_{s\ell} \\
&= -E_j \left[g \left(D_{E_k}(X^s E_s), E_\ell \right) \right] + g \left(D_{E_k}(X^s E_s), D_{E_j} E_\ell \right) + g \left(D_{E_j} E_k, D_{E_\ell}(X^s E_s) \right) \\
&\quad - X^s \mathcal{R}_{jsk\ell} - (\Gamma^s_{jk})' \omega_{s\ell} \\
&= -E_j \left[g \left(X^s_{;k} E_s + (D_{E_k}(X^s E_s))^\perp, E_\ell \right) \right] + g \left(X^s_{;k} E_s + (D_{E_k}(X^s E_s))^\perp, D_{E_j} E_\ell \right) \\
&\quad + g \left(D_{E_j} E_k, X^s_{;\ell} E_s + (D_{E_\ell}(X^s E_s))^\perp \right) - X^s \mathcal{R}_{jsk\ell} - (\Gamma^s_{jk})' \omega_{s\ell}.
\end{aligned}$$

Now, using the fact that we have arranged to have $D_{E_j} E_k$ orthogonal to Σ at p_0 at $t = 0$, we can deduce

$$
\begin{aligned}
(B_{jk\ell})' &= -E_j \left[g \left(X^s_{;k} E_s, E_\ell \right) \right] + g \left((D_{E_k}(X^s E_s))^\perp, D_{E_j} E_\ell \right) \\
&\quad + g \left(D_{E_j} E_k, (D_{E_\ell}(X^s E_s))^\perp \right) - X^s \mathcal{R}_{jsk\ell} - (\Gamma^s_{jk})' \omega_{s\ell} \\
&= -X_{\ell;kj} + X^s g \left((D_{E_k} E_s)^\perp, (D_{E_j} E_\ell)^\perp \right) + X^s g \left((D_{E_j} E_k)^\perp, (D_{E_\ell} E_s)^\perp \right) \\
&\quad - X^s \mathcal{R}_{jsk\ell} - (\Gamma^s_{jk})' \omega_{s\ell}.
\end{aligned}
\tag{15}
$$

To deal with the $(D_{E_j} E_k)^\perp$ terms, we introduce the operator $\mathcal{D}$ on $T_p M$ which is the difference between the orthogonal projection onto $N_p \Sigma$ and the orthogonal projection onto $J(T_p \Sigma)$. Now, for any $W \in N_p \Sigma$, we can write

$$
W = h^{ij} g(W, J E_j) J E_i + \mathcal{D}(W) = -h^{ij} \omega(W, E_j) J E_i + \mathcal{D}(W).
$$

where used the fact that J is an isometry. Consequently (15) becomes

$$
\begin{aligned}
(B_{jk\ell})' &= -X_{\ell;kj} + X^s h^{uq} B_{ksq} B_{j\ell u} + X^s h^{uq} B_{\ell sq} B_{jku} \\
&\quad + X^s g \left(\mathcal{D}((D_{E_k} E_s)^\perp), (D_{E_j} E_\ell)^\perp \right) + X^s g \left((D_{E_j} E_k)^\perp, \mathcal{D}((D_{E_\ell} E_s)^\perp) \right) \\
&\quad - X^s \mathcal{R}_{jsk\ell} - (\Gamma^s_{jk})' \omega_{s\ell}.
\end{aligned}
$$

We have now computed all the separate constituents of $L^{(2)}_\rho(X)$. It remains only to put everything together. We find

$$
\begin{aligned}
L^{(2)}_\rho(X) &= (h^{\ell m})' H_{\ell;m} - h^{\ell m} (\Gamma^s_{\ell m})' H_s + h^{\ell m} \left((H_\ell)' \right)_{;m} \\
&= -h^{\ell u} h^{mq} (h^{jk} B_{jk\ell})_{;m} \left(\beta(X)_{uq} + \mathcal{C}(X)_{uq} \right) \\
&\quad - \tfrac{1}{2} h^{\ell m} h^{jk} h^{sq} B_{jks} \left(\beta(X)_{q\ell;m} + \beta(X)_{qm;\ell} - \beta(X)_{\ell m;q} \right) \\
&\quad - \tfrac{1}{2} h^{\ell m} h^{jk} h^{sq} B_{jks} \left(\mathcal{C}(X)_{q\ell;m} + \mathcal{C}(X)_{qm;\ell} - \mathcal{C}(X)_{\ell m;q} \right) \\
&\quad - \left(h^{\ell m} h^{ju} h^{kq} B_{jk\ell} (\beta(X)_{uq} + \mathcal{C}(X)_{uq}) \right)_{;m} \\
&\quad + h^{\ell m} h^{jk} \left(-X_{\ell;kj} + X^s h^{uq} B_{ksq} B_{j\ell u} + X^s h^{uq} B_{\ell sq} B_{jku} \right)_{;m} \\
&\quad + h^{\ell m} h^{jk} X^s \left(g \left(\mathcal{D}((D_{E_k} E_s)^\perp), (D_{E_j} E_\ell)^\perp \right) + g \left((D_{E_j} E_k)^\perp, \mathcal{D}((D_{E_\ell} E_s)^\perp) \right) \right)_{;m} \\
&\quad - h^{\ell m} h^{jk} \left(X^s \mathcal{R}_{jsk\ell} + (\Gamma^s_{jk})' \omega_{s\ell} \right)_{;m} = \mathcal{E}_1(X) + \mathcal{E}_2(X)
\end{aligned}
$$

where $\mathcal{E}_1(X)$ and $\mathcal{E}_2(X)$ are as in the statement of the proposition. In attaining these expressions, we have expanded $\beta(X)_{ij} = X^s (B_{isj} + B_{jsi})$. The point of arranging the outcome of the calculation in this way is because the term $\mathcal{E}_1(X)$ has the same form as the linearization of the Hamiltonian stationary Lagrangian differential operator at a Lagrangian submanifold while the term $\mathcal{E}_2(X)$ vanishes at a Lagrangian submanifold. $\qquad \square$

Thus we can write $L_\rho(X) = \left(L^{(1)}_\rho(X), L^{(2)}_\rho(X) \right) = (d(X \lrcorner \omega_\rho), \mathcal{E}_1(X) + \mathcal{E}_2(X))$. We express $L_\rho(X)$ as the decomposition $L^{(s)}_\rho(X) = \mathring{L}^{(s)}(X) + P^{(s)}_\rho(X)$ for $s = 1, 2$. Of course, $L^{(1)}_\rho(X) = d(X \lrcorner \omega_\rho)$, and so

$$
P^{(1)}_\rho(X) = d(X \lrcorner \omega_\rho) - d(X \lrcorner \mathring{\omega}) = d(X \lrcorner (\omega_\rho - \mathring{\omega})).
$$

For $P_\rho^{(2)}(X)$, observe that $\mathcal{E}_1(X)$ has the same form as $\mathring{L}^{(2)}(X)$ and $\mathcal{E}_2(X)$ vanishes when $\rho = 0$. Thus formally we can decompose

$$P_\rho^{(2)}(X) = \left(\mathcal{E}_1(X) - \mathring{L}^{(2)}(X)\right) + \mathcal{E}_2(X).$$

We will not determine the precise form of the operator $\mathcal{E}_1(X) - \mathring{L}^{(2)}(X)$ since these details will not be needed in the sequel.

Corollary 9 *The components of the operator P_ρ are*

$$P_\rho^{(1)}(X) : = \mathrm{d}(X \lrcorner (\omega_\rho - \mathring{\omega}))$$
$$P_\rho^{(2)}(X) : = \left(\mathcal{E}_1(X) - \mathring{L}^{(2)}(X)\right) + \mathcal{E}_2(X)$$

with notation as in Proposition 8.

We now obtain a corresponding decomposition $L_\rho^{(s)} := \mathring{L}^{(s)} + P_\rho^{(s)}$ where $P_\rho^{(s)} := P_\rho^{(s)} \circ \mathcal{X}$.

4.5 Estimates for the perturbed linearization

We work on a fixed ball $\mathcal{B} \subset \mathbb{C}^2$ around the origin, compactly contained inside the domain of the Kähler potential F_ρ of the metric g_ρ. The $C^{k,\alpha}$ norms will be taken with respect to the background metric $\mathring{g}$ when the quantity being estimated is defined in $\mathbb{C}^2$ and with respect to the induced metric $\mathring{h}$ when the quantity being estimated is defined on a submanifold Σ. Note that these norms are equivalent to those defined by the metric g_ρ and its induced metric h on Σ. We begin with the following lemma. For simplicity of notation, we do not use a subscript for quantities such as h, B and ∇ induced by g_ρ.

Lemma 10 *Let Σ be a totally real submanifold of $\mathcal{B}$ equipped with the Kähler metric g_ρ with Kähler potential F_ρ as in (7). Fix $\alpha \in (0, 1)$ and $k \in \mathbb{N}$. There is a constant C independent of ρ so that for all $X \in \Gamma(J(T\Sigma))$ and $W \in \Gamma(N\Sigma)$ the following estimates hold:*

$$\|g_\rho - \mathring{g}\|_{C^{k,\alpha}(\mathcal{B})} \leq C\rho^2 \qquad \|\nabla \cdot X - \mathring{\nabla} \cdot X\|_{C^{k-1,\alpha}(\Sigma)} \leq C\rho^2 \|X\|_{C^{k-1,\alpha}(\Sigma)}$$
$$\|\omega_\rho - \mathring{\omega}\|_{C^{k,\alpha}(\mathcal{B})} \leq C\rho^2 \qquad \|\mathcal{C}(X)\|_{C^{k-1,\alpha}(\Sigma)} \leq C\rho^2 \|X\|_{C^{k,\alpha}(\Sigma)}$$
$$\|B - \mathring{B}\|_{C^{k-1,\alpha}(\Sigma)} \leq C\rho^2 \qquad \|\mathcal{D}(W)\|_{C^{k,\alpha}(\Sigma)} \leq C\rho^2 \|W\|_{C^{k,\alpha}(\Sigma)}$$
$$\|H - \mathring{H}\|_{C^{k-1,\alpha}(\Sigma)} \leq C\rho^2 \qquad \|\mathcal{E}_2(X)\|_{C^{k-3,\alpha}(\Sigma)} \leq C\rho^2 \|X\|_{C^{k,\alpha}(\Sigma)}.$$

Furthermore, the operator $\mathcal{D}$ vanishes if Σ is Lagrangian.

Proof The estimates mostly follow from the estimate of the Kähler potential $F_\rho(z, \bar{z}) := \frac{1}{2}\|z\|^2 + \rho^2 \hat{F}_\rho(z, \bar{z})$, where $\hat{F}_\rho(z, \bar{z}) := \rho^{-4}\hat{F}(\rho z, \rho \bar{z})$. Recall that on $\mathcal{B}$, for any multi-index α the derivative $\frac{\partial^\alpha \hat{F}}{\partial \zeta^{\alpha_1} \partial \bar{\zeta}^{\alpha_2}}(\zeta, \bar{\zeta})$ is $\mathcal{O}(\|\zeta\|^{4-\alpha})$ for $|\alpha| \leq 4$, and $\mathcal{O}(1)$ for $|\alpha| > 4$. This immediately gives the first two estimates. The estimate on the symplectic second fundamental form comes from the following (and then immediately implies the estimate on the mean curvature one-form):

$$B(X, Y, Z) - \mathring{B}(X, Y, Z) = \omega_\rho((D_X Y)^\perp, Z) - \mathring{\omega}((\mathring{D}_X Y)^\perp, Z),$$

where $(D_X Y)^\perp = D_X Y - h^{ij} g_\rho(D_X Y, E_j) E_i$ and $(\mathring{D}_X Y)^\perp = \mathring{D}_X Y - \mathring{h}^{ij} \mathring{g}(\mathring{D}_X Y, E_j) E_i$. The above estimate of $(g_\rho - \mathring{g})$ yields the analogous estimate of $D - \mathring{D}$, which together with the equation above then yields the estimate of $B - \mathring{B}$, as well as the estimate on the divergence.

We now estimate $\mathcal{D}$, which, together with the above estimates, will also yield the estimate of $\mathcal{E}_2$, and thus complete the proof. Let $W \in N_p\Sigma$ be a unit vector. Recall from above that

$$\mathcal{D}(W) = W - h^{ij}g_\rho(W, JE_j)JE_i = W + h^{ij}\omega_\rho(W, E_j)JE_i.$$

If we use the orthogonal decomposition of W with respect to the metric $\mathring{g}$, denoting it as $W = \mathring{W}^\parallel + \mathring{W}^\perp$, then since $g_\rho(W, E_j) = 0$, we have immediately $\mathring{g}(W, E_j) = \mathcal{O}(\rho^2)$. Thus $\mathring{W}^\parallel = O(\rho^2)$. Furthermore, since Σ is Lagrangian for $\mathring{\omega}$, then $\mathring{W}^\perp = -\mathring{h}^{ij}\mathring{\omega}(\mathring{W}^\perp, E_j)JE_i = -\mathring{h}^{ij}\mathring{\omega}(W, E_j)JE_i$. Thus $\mathcal{D}(W) - \mathring{W}^\parallel = \mathring{W}^\perp + h^{ij}\omega_\rho(W, E_j)JE_i = \mathcal{O}(\rho^2)$. □

Based on these elementary estimates, we have the following estimates of P_ρ and $\boldsymbol{P}_\rho$ on a totally real submanifold Σ.

Proposition 11 *Let Σ be a totally real submanifold of $\mathcal{B}$ equipped with the Kähler metric g_ρ. Fix $k \in \mathbb{N}$ and $\alpha \in (0, 1)$. There is a constant C independent of ρ so that (with norms computed on Σ)*

$$\|P_\rho^{(1)}(X)\|_{C^{k,\alpha}} \leq C\rho^2 \|X\|_{C^{k+1,\alpha}}$$

$$\|P_\rho^{(2)}(X)\|_{C^{k,\alpha}} \leq C\rho^2 \|X\|_{C^{k+3,\alpha}}$$

$$\|\boldsymbol{P}_\rho^{(1)}(u, v)\|_{C^{k,\alpha}} \leq C\rho^2 \|(u, v)\|_{C^{k+2,\alpha} \times C^{k+2,\alpha}}$$

$$\|\boldsymbol{P}_\rho^{(2)}(u, v)\|_{C^{k,\alpha}} \leq C\rho^2 \|(u, v)\|_{C^{k+4,\alpha} \times C^{k+4,\alpha}}.$$

5 Solving the Hamiltonian stationary Lagrangian PDE

5.1 Outline

In this final section of the paper, the equation $\boldsymbol{\Phi}_\rho(u, v) = (0, 0)$ will be solved for all ρ sufficiently small using a perturbative technique. An initial difficulty that must be overcome is that it is not possible to find a suitable inverse for the linearized operator $\boldsymbol{L}_\rho := \mathrm{D}_{(0,0)}\boldsymbol{\Phi}_\rho$ with ρ-independent norm because the operator $\mathring{\boldsymbol{L}} := \mathrm{D}_{(0,0)}\mathring{\boldsymbol{\Phi}}$ has a non-trivial, six-dimensional kernel and fails to be surjective since its adjoint has a seven-dimensional kernel. This fact makes a three-step approach for solving $\boldsymbol{\Phi}_\rho(u, v) = (0, 0)$ necessary. In what follows, we identify two-forms on Σ_r with functions via the Hodge star operator.

Step 1 The first step is to solve a *projected* problem wherein the difficulties engendered by the kernel and co-kernel of $\mathring{\boldsymbol{L}}$ are avoided. Let $\mathcal{K}$ be the kernel of $\mathring{\boldsymbol{L}}$ and let $\mathcal{K}^*$ be the kernel of $\mathring{\boldsymbol{L}}^*$. Let

$$\pi : C^{2,\alpha}(\mathrm{d}\Lambda^1(\Sigma_r)) \times C^{0,\alpha}(\Sigma_r) \to \left(C^{2,\alpha}(\mathrm{d}\Lambda^1(\Sigma_r)) \times C^{0,\alpha}(\Sigma_r)\right) \cap [\mathcal{K}^*]^\perp$$

be the L^2-orthogonal projection onto $[\mathcal{K}^*]^\perp$ with respect to the volume measure induced from the Euclidean ambient metric, and consider the operator (defined near $(0, 0)$)

$$\pi \circ \boldsymbol{\Phi}_\rho\big|_{\mathcal{K}^\perp} : \left(C_0^{4,\alpha}(\Sigma_r) \times C_0^{4,\alpha}(\Sigma_r)\right) \cap \mathcal{K}^\perp \to \left(C^{2,\alpha}(\mathrm{d}\Lambda^1(\Sigma_r)) \times C^{0,\alpha}(\Sigma_r)\right) \cap [\mathcal{K}^*]^\perp.$$

The first step is thus to solve $\pi \circ \boldsymbol{\Phi}_\rho\big|_{\mathcal{K}^\perp}(u, v) = (0, 0)$. The linearization of this new operator is $\pi \circ \boldsymbol{L}_\rho\big|_{\mathcal{K}^\perp}$ which is by definition invertible at $\rho = 0$. This operator remains invertible for sufficiently small $\rho > 0$, and it will be shown below that a solution of the non-linear problem

$$\pi \circ \mathbf{\Phi}_\rho \big|_{\mathcal{K}^\perp} (u, v) = (0, 0)$$

can be found. We will denote the solution by (u_ρ, v_ρ) and let $\tilde{\Sigma}_r(\mathcal{U}_p) := \mu_{\mathcal{X}(u_\rho, v_\rho)}(\Sigma_r(\mathcal{U}_p))$ be the perturbed submanifold generated by this solution; we will abbreviate this by $\tilde{\Sigma}_r$ when there is no cause for confusion.

Step 2 The previous step shows that a solution $(u, v) := (u_\rho, v_\rho)$ of the projected problem on Σ_r can always be found so long as ρ is sufficiently small. One should realize that the solution (u_ρ, v_ρ) that has been found depends implicitly on the point $p \in M$ and the choice of unitary frame $\mathcal{U}_p$ at p out of which Σ_r has been constructed. Moreover, this dependence is smooth as a standard consequence of the fixed-point argument used to find (u_ρ, v_ρ). The solution is such that $\mathbf{\Phi}_\rho(u_\rho, v_\rho)$ is an *a priori* non-trivial but small quantity that belongs to $\mathcal{K}^*$.

In the second step of the proof of the Main Theorem, it will be shown that when an *existence condition* is satisfied at the point $p \in M$, there exists p' near p and a frame $\mathcal{U}_{p'}$ so that $\mathbf{\Phi}_\rho(u_\rho, v_\rho)$ vanishes except for a component in the space $\mathrm{span}_\mathbb{R}\{(0, 1)\}$. We set this up as follows. First, write $\mathcal{K}^* = \mathrm{span}_\mathbb{R}\{(0, 1)\} \oplus \mathcal{K}_0^*$ where $\mathcal{K}_0^* := \mathrm{span}_\mathbb{R}\{f^{(1)}w^{(1)}, \ldots, f^{(6)}w^{(6)}\}$, with the functions $f^{(j)}$ and the constant vectors $w^{(j)}$ determined in Corollary 7. Therefore

$$\mathbf{\Phi}_\rho(u_\rho, v_\rho) = a(0, 1) + \sum_{j=1}^{6} b_j f^{(j)} w^{(j)} \quad \text{for some} \quad a, b_1, \ldots, b_6 \in \mathbb{R}.$$

Now define a smooth mapping $G_\rho : U(M) \to \mathbb{R}^6 \approx \mathbb{C}^3$ on the unitary 2-frame bundle $U(M)$ over M, given by

$$G_\rho(\mathcal{U}_p) := \left(I_\rho^{(1)}(\mathcal{U}_p), \ldots, I_\rho^{(6)}(\mathcal{U}_p) \right)$$
$$\approx \left(I_\rho^{(1)}(\mathcal{U}_p) + \mathrm{i} I_\rho^{(2)}(\mathcal{U}_p), I_\rho^{(3)}(\mathcal{U}_p) + \mathrm{i} I_\rho^{(4)}(\mathcal{U}_p), I_\rho^{(5)}(\mathcal{U}_p) + \mathrm{i} I_\rho^{(6)}(\mathcal{U}_p) \right) \in \mathbb{C}^3$$

where

$$I_\rho^{(j)}(\mathcal{U}_p) := \int_{\Sigma_r} \left(f^{(j)} - c^{(j)} \right) w^{(j)} \cdot \mathbf{\Phi}_\rho(u_\rho, v_\rho) \mathrm{dVol}_{\Sigma_r} \tag{16}$$

and $c^{(i)}$ has been chosen to ensure that $\int_{\Sigma_r} \left(f^{(j)} - c^{(j)} \right) \mathrm{dVol}_{\Sigma_r} = 0$. We have

$$I_\rho^{(j)}(\mathcal{U}_p) = \sum_{k=1}^{6} b_k (w^{(j)} \cdot w^{(k)}) \int_{\Sigma_r} (f^{(j)} - c^{(j)}) f^{(k)} \mathrm{dVol}_{\Sigma_r}$$
$$= \sum_{k=1}^{6} b_k (w^{(j)} \cdot w^{(k)}) \int_{\Sigma_r} (f^{(j)} - c^{(j)})(f^{(k)} - c^{(k)}) \mathrm{dVol}_{\Sigma_r}.$$

We would now like to find $\mathcal{U}_p$ so that $G_\rho(\mathcal{U}_p) \equiv 0$. This will turn imply that $b_k = 0$ for all k, because the matrix whose entries are $\int_{\Sigma_r} (f^{(j)} - c^{(j)}) w^{(j)} \cdot (f^{(k)} - c^{(k)}) w^{(k)} \mathrm{dVol}_{\Sigma_r}$ is invertible. This holds because $(f^{(j)} - c^{(j)}) w^{(j)}$ for $j = 1, \ldots 6$, forms an independent set; one can also argue this by perturbation, since the matrix $\int_{\Sigma_r} f^{(j)} f^{(k)} \mathrm{dVol}_{\Sigma_r}^\circ$ is diagonal and invertible.

The idea for locating a zero of G_ρ is first to find $\mathcal{U}_p$ so that $G_\rho(\mathcal{U}_p)$ vanishes to lowest order in a Taylor expansion in powers of ρ, but in such a way that G_ρ remains locally surjective at this $\mathcal{U}_p$. The inverse function theorem for finite-dimensional manifolds can then be invoked to find a nearby $\mathcal{U}_{p'}$ for which $G_\rho(\mathcal{U}_{p'}) \equiv 0$ exactly.

Step 3 From the previous step, we now have $\mathcal{U}_{p'}$ and (u_ρ, v_ρ) so that $\boldsymbol{\Phi}_\rho(u_\rho, v_\rho) = (0, a)$. In other words, the associated surface $\tilde{\Sigma}_r$ built from the corresponding unitary frame and deformation vector field satisfies $\nabla \cdot H\left(\tilde{\Sigma}_r\right) = a$. But the divergence theorem can now be invoked to show that $a = 0$, thereby completing the proof of the Main Theorem.

Remark Unless otherwise noted, the norms used below are all taken on Σ_r.

5.2 Estimates for the approximate solution

To begin, we must compute the size of $\|\boldsymbol{\Phi}_\rho(0, 0)\|_{C^{2,\alpha} \times C^{0,\alpha}}$, which must be sufficiently small for the perturbation method of Step 1 to succeed. We assume $\Sigma_r \subset \mathcal{B}$, as in Lemma 10.

Proposition 12 *There is a constant $C > 0$ independent of ρ so that*

$$\|\boldsymbol{\Phi}_\rho(0, 0)\|_{C^{2,\alpha} \times C^{0,\alpha}} \leq C\rho^2.$$

Proof By Lemma 1, we have $\mathring{\boldsymbol{\Phi}}(0, 0) = (0, 0)$. By Lemma 10, we have $\|\omega_\rho - \mathring{\omega}\|_{C^{2,\alpha}(\mathcal{B})} \leq C\rho^2$. Furthermore, by writing

$$\nabla \cdot H = \mathring{\nabla} \cdot \mathring{H} + (\nabla - \mathring{\nabla}) \cdot \mathring{H} + \nabla \cdot (H - \mathring{H}) = (\nabla - \mathring{\nabla}) \cdot \mathring{H} + \nabla \cdot (H - \mathring{H}),$$

we have

$$\|\nabla \cdot H\|_{C^{0,\alpha}} \leq C\rho^2 \|\mathring{H}\|_{C^{1,\alpha}} + \|H - \mathring{H}\|_{C^{1,\alpha}} \leq C\rho^2$$

again using the estimates of Lemma 10. $\qquad\square$

5.3 Solving the projected problem

This section proves that Step 1 from the outline above can be carried out.

Theorem 13 *For all ρ sufficiently small, there exists $(u_\rho, v_\rho) \in \left(C_0^{4,\alpha}(\Sigma_r) \times C_0^{4,\alpha}(\Sigma_r)\right) \cap \mathcal{K}^\perp$ that satisfies*

$$\pi \circ \boldsymbol{\Phi}_\rho(u_\rho, v_\rho) = (0, 0).$$

Moreover, the estimate $\|(u_\rho, v_\rho)\|_{C^{4,\alpha} \times C^{4,\alpha}} \leq C\rho^2$ holds.

Proof The solvability of the equation $\pi \circ \boldsymbol{\Phi}_\rho(u, v) = (0, 0)$ is governed by the behaviour of the linearized operator $\pi \circ L_\rho$ between the Banach spaces given in the statement of the theorem, as well as on the size of $\|\boldsymbol{\Phi}_\rho(0, 0)\|_{C^{2,\alpha} \times C^{0,\alpha}}$, which we know to be $\mathcal{O}(\rho^2)$ by Proposition 12.

First, by standard elliptic theory, the operator $\pi \circ \mathring{L}$ is invertible between $\mathcal{K}^\perp$ and $[\mathcal{K}^*]^\perp$ with the estimate

$$\|\pi \circ \mathring{L}(u, v)\|_{C^{2,\alpha} \times C^{0,\alpha}} \geq C\|(u, v)\|_{C^{4,\alpha} \times C^{4,\alpha}}$$

where C is a constant independent of ρ. Consequently, if ρ is sufficiently small, then the operator $\pi \circ L_\rho$ is uniformly injective with the estimate

$$\|\pi \circ L_\rho(u, v)\|_{C^{2,\alpha} \times C^{0,\alpha}} \geq \frac{C}{2}\|(u, v)\|_{C^{4,\alpha} \times C^{4,\alpha}}.$$

Hence by perturbation, the operator $\pi \circ L_\rho$ is also surjective onto $[\mathcal{K}^*]^\perp$ and the inverse is bounded above independently of ρ.

The remainder of the proof uses the contraction mapping theorem. First, we note that we will work in a convex neighborhood of $(0, 0)$ inside the domain of $\boldsymbol{\Phi}_\rho$, and so that for (u, v) in this neighborhood, we have $\mu_{\mathcal{X}(u,v)}(\Sigma_r(\mathcal{U}_p)) \subset \mathcal{B}$, so that the estimates of Lemma 10 hold. We write

$$\pi \circ \boldsymbol{\Phi}_\rho(u, v) := \pi \circ \boldsymbol{\Phi}_\rho(0, 0) + \pi \circ \boldsymbol{L}_\rho(u, v) + \pi \circ \boldsymbol{Q}_\rho(u, v)$$

where $\boldsymbol{Q}_\rho$ is the quadratic (in u and v) remainder of $\boldsymbol{\Phi}_\rho$. It is fairly straightforward to show that $\boldsymbol{Q}_\rho$ satisfies the estimate

$$\|\boldsymbol{Q}_\rho(u_1, v_1) - \boldsymbol{Q}_\rho(u_2, v_2)\|_{C^{2,\alpha} \times C^{0,\alpha}}$$
$$\leq C(\|(u_1, v_1)\|_{C^{4,\alpha} \times C^{4,\alpha}} + \|(u_2, v_2)\|_{C^{4,\alpha} \times C^{4,\alpha}})\|(u_1 - u_2, v_1 - v_2)\|_{C^{4,\alpha} \times C^{4,\alpha}}$$

for some constant C independent of ρ, provided ρ is sufficiently small. This follows by perturbation because such an estimate is certainly true for the quadratic remainder of $\mathring{\boldsymbol{\Phi}}$. Now let $\boldsymbol{L}_\rho^{-1} : [\mathcal{K}^*]^\perp \to \mathcal{K}^\perp$ denote the inverse of $\boldsymbol{L}_\rho$ onto $\mathcal{K}^\perp$. By proposing the *Ansatz* $(u, v) := -\boldsymbol{L}_\rho^{-1}\left((w, \xi) + \pi \circ \boldsymbol{\Phi}_\rho(0, 0)\right)$, for $(w, \xi) \in [\mathcal{K}^*]^\perp$, the equation $\pi \circ \boldsymbol{\Phi}_\rho(u, v) = (0, 0)$ becomes equivalent to the fixed-point problem for the map

$$\mathcal{N}_\rho : (w, \xi) \mapsto \pi \circ \boldsymbol{Q}_\rho\left(-\boldsymbol{L}_\rho^{-1}\left((w, \xi) + \pi \circ \boldsymbol{\Phi}_\rho(0, 0)\right)\right)$$

on $[\mathcal{K}^*]^\perp$. For small enough ρ, the non-linear mapping $\mathcal{N}_\rho$ verifies the estimates required to find a fixed point in a closed ball $B \subset [\mathcal{K}^*]^\perp$ of radius equal to $\|\boldsymbol{\Phi}_\rho(0, 0)\|_{C^{2,\alpha} \times C^{0,\alpha}} = \mathcal{O}(\rho^2)$, by virtue of the ρ-independent estimates that have been found for $\boldsymbol{L}_\rho^{-1}$ and $\boldsymbol{Q}_\rho$. For example, for $(w, \xi) \in B$,

$$\|\mathcal{N}_\rho(w, \xi)\|_{C^{2,\alpha} \times C^{0,\alpha}} \leq C\|\boldsymbol{\Phi}_\rho(0, 0)\|_{C^{2,\alpha} \times C^{0,\alpha}}^2 \leq \|\boldsymbol{\Phi}_\rho(0, 0)\|_{C^{2,\alpha} \times C^{0,\alpha}}$$

for ρ small enough; hence the set B is mapped to itself under $\mathcal{N}_\rho$. Furthermore, $\mathcal{N}_\rho$ is a contraction on B as a result of the bilinear estimate on $\boldsymbol{Q}_\rho$ given above. Consequently, $\mathcal{N}_\rho$ must have a fixed point $(w, \xi) \in B$ which thus satisfies $\|(w, \xi)\|_{C^{2,\alpha} \times C^{0,\alpha}} \leq C\rho^2$ for some constant C independent of ρ. The desired estimate follows. $\qquad\square$

Remark The solution (u_ρ, v_ρ) is in fact smooth by elliptic regularity theory, and the estimate $\|(u_\rho, v_\rho)\|_{C^{k,\alpha} \times C^{k,\alpha}} \leq C(k)\rho^2$ holds for all $k \in \mathbb{N}$, where $C(k)$ is independent of ρ.

5.4 Derivation of the existence condition

The remainder of the proof begins with a more careful investigation of the integrals $I_\rho^{(i)}(\mathcal{U}_p)$ making up the projection map $G_\rho : U(M) \to \mathbb{R}^6$ of (16). We can relate these integrals to the ambient geometry of M to lowest order in ρ using the first variation formula along with Stokes' theorem. We let $\mathring{\nabla} f$ and ∇f be the gradient vectors of f on Σ_r in the respective metrics. Define $w_1^{(j)}, w_2^{(j)}$ as the first and second components of $w^{(j)}$, respectively.

Lemma 14 *The following formula holds.*

$$I_\rho^{(j)}(\mathcal{U}_p) = w_2^{(j)} \left(\int_{\Sigma_r} \mathring{g}(J\mathring{\nabla} f^{(j)}, \vec{H}_{\Sigma_r} - \vec{H}_{\Sigma_r}^\circ) \, d\text{Vol}_{\Sigma_r}^\circ \right.$$

$$\left. - \int_{\Sigma_r} \mathring{H}(\Sigma_r) \left(\mathring{\nabla} f^{(j)} \right) \left(d\text{Vol}_{\Sigma_r} - d\text{Vol}_{\Sigma_r}^\circ \right) \right)$$

$$+ w_1^{(j)} \int_{\Sigma_r} f^{(j)} \cdot (\omega_\rho - \mathring{\omega}) + \mathcal{O}(\rho^4). \tag{17}$$

Proof With a direct computation, we find

$$\int_{\Sigma_r} (f^{(j)} - c^{(j)}) \mathbf{\Phi}_\rho(u_\rho, v_\rho) \cdot w^{(j)} d\text{Vol}_{\Sigma_r}$$

$$= w_2^{(j)} \int_{\Sigma_r} \nabla \cdot H(\Sigma_r)(f^{(j)} - c^{(j)}) d\text{Vol}_{\Sigma_r} + w_1^{(j)} \int_{\Sigma_r} (f^{(j)} - c^{(j)})(\omega_\rho - \mathring{\omega})$$

$$+ \int_{\Sigma_r} (f^{(j)} - c^{(j)})\mathring{\mathbf{L}}(u_\rho, v_\rho) \cdot w^{(j)} d\text{Vol}_{\Sigma_r}^\circ$$

$$+ \int_{\Sigma_r} (f^{(j)} - c^{(j)}) \mathbf{L}_\rho(u_\rho, v_\rho) \cdot w^{(j)} \left(d\text{Vol}_{\Sigma_r} - d\text{Vol}_{\Sigma_r}^\circ \right)$$

$$+ \int_{\Sigma_r} (f^{(j)} - c^{(j)}) \mathbf{P}_\rho(u_\rho, v_\rho) \cdot w^{(j)} d\text{Vol}_{\Sigma_r}^\circ + \int_{\Sigma_r} (f^{(j)} - c^{(j)}) \mathbf{Q}_\rho(u_\rho, v_\rho) \cdot w^{(j)} d\text{Vol}_{\Sigma_r}$$

$$= -w_2^{(j)} \int_{\Sigma_r} H(\Sigma_r) \left(\nabla f^{(j)} \right) d\text{Vol}_{\Sigma_r} + w_1^{(j)} \int_{\Sigma_r} f^{(j)} \cdot (\omega_\rho - \mathring{\omega})$$

$$+ \int_{\Sigma_r} (f^{(j)} - c^{(j)})\mathring{\mathbf{L}}(u_\rho, v_\rho) \cdot w^{(j)} d\text{Vol}_{\Sigma_r}^\circ + \mathcal{O}(\rho^4).$$

Here we have used the expansion $\mathbf{\Phi}_\rho(u_\rho, v_\rho) = \mathbf{\Phi}_\rho(0, 0) + \mathbf{L}_\rho(u_\rho, v_\rho) + \mathbf{Q}_\rho(u_\rho, v_\rho)$, where $\mathbf{L}_\rho = \mathring{\mathbf{L}} + \mathbf{P}_\rho$ and $\mathbf{Q}_\rho$ is the quadratic remainder of the operator $\mathbf{\Phi}_\rho$, along with Stokes' theorem, as well as $\mathring{\omega}|_{\Sigma_r} = 0$ and the following facts:

- $\|(u_\rho, v_\rho)\|_{C^{4,\alpha} \times C^{4,\alpha}}$, $\|\mathbf{L}_\rho(u_\rho, v_\rho)\|_{C^0}$ and $\|\omega_\rho - \omega\|_{C^0}$ are all $\mathcal{O}(\rho^2)$.
- $\|\mathbf{P}_\rho(u_\rho, v_\rho)\|_{C^0} \le C\rho^2 \|(u_\rho, v_\rho)\|_{C^{4,\alpha} \times C^{4,\alpha}} = \mathcal{O}(\rho^4)$.
- $\|\mathbf{Q}_\rho(u_\rho, v_\rho)\|_{C^0} \le C\|(u_\rho, v_\rho)\|_{C^{4,\alpha} \times C^{4,\alpha}}^2 = \mathcal{O}(\rho^4)$.
- the difference between the volume forms appearing above is $\mathcal{O}(\rho^2)$.
- $\int_{\Sigma_r} f^{(i)} d\text{Vol}_{\Sigma_r}^\circ = 0$ which implies $|c^{(i)}| = \mathcal{O}(\rho^2)$.

To complete the proof of the lemma, we continue as follows. Note that since $(f^{(j)} - c^{(j)})w^{(j)}$ belongs to the kernel of $\mathring{\mathbf{L}}^*$, then $\int_{\Sigma_r} (f^{(j)} - c^{(j)})\mathring{\mathbf{L}}(u_\rho, v_\rho) \cdot w^{(j)} d\text{Vol}_{\Sigma_r}^\circ = 0$. Furthermore, we let $\vec{H}_{\Sigma_r}$ and $\vec{H}_{\Sigma_r}^\circ$ denote the normal vector-valued mean curvatures of Σ_r with respect to the metric g_ρ and the Euclidean metric, respectively. We let E_1, E_2 be a local $\mathring{g}$-orthonormal tangent frame on Σ_r, and we write (summing over repeated indices) $\vec{H}_{\Sigma_r} = Z^s E_s + \hat{Z}^s J E_s$,

where Z^s and $\hat{Z}^s$ are real-valued functions, so that $J\vec{H}_{\Sigma_r} = Z^s J E_s - \hat{Z}^s E_s$. Furthermore, $Z^s E_s = [\vec{H}_{\Sigma_r}]^{\|0} = [\vec{H}_{\Sigma_r} - \vec{H}^\circ_{\Sigma_r}]^{\|0} = \mathcal{O}(\rho^2)$. If h is the metric induced from g_ρ on Σ_r, we have $\nabla f = h^{k\ell} E_\ell(f) E_k$ and $\mathring{\nabla} f = \delta^{k\ell} E_\ell(f) E_k$. Also, since $(g_\rho - \mathring{g}) = \mathcal{O}(\rho^2)$, we have

$$
\begin{aligned}
-g_\rho(J\nabla f, \vec{H}_{\Sigma_r}) &= h^{k\ell} E_\ell(f) g_\rho \left(E_k, Z^s J E_s - \hat{Z}^s E_s \right) \\
&= h^{k\ell} E_\ell(f) \left(g_\rho(E_k, J E_s) Z^s - h_{ks} \hat{Z}^s \right) \\
&= -E_\ell(f)\hat{Z}^\ell + h^{k\ell} E_\ell(f) Z^s g_\rho(E_k, J E_s) \\
&= \mathring{g}(\mathring{\nabla} f, J\vec{H}_{\Sigma_r}) + h^{k\ell} E_\ell(f) Z^s (g_\rho - \mathring{g})(E_k, J E_s) \\
&= -\mathring{g}(J\mathring{\nabla} f, \vec{H}_{\Sigma_r}) + \mathcal{O}(\rho^4).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_{\Sigma_r} H(\Sigma_r) \left(\nabla f^{(j)} \right) d\mathrm{Vol}_{\Sigma_r} &= -\int_{\Sigma_r} g_\rho(J\nabla f^{(j)}, \vec{H}_{\Sigma_r}) d\mathrm{Vol}_{\Sigma_r} \\
&= -\int_{\Sigma_r} \mathring{g}(J\mathring{\nabla} f^{(j)}, \vec{H}_{\Sigma_r}) d\mathrm{Vol}_{\Sigma_r} + \mathcal{O}(\rho^4) \\
&= -\int_{\Sigma_r} \mathring{g}(J\mathring{\nabla} f^{(j)}, \vec{H}_{\Sigma_r} - \vec{H}^\circ_{\Sigma_r}) \, d\mathrm{Vol}_{\Sigma_r} \\
&\quad - \int_{\Sigma_r} \mathring{g}(J\mathring{\nabla} f^{(j)}, \vec{H}^\circ_{\Sigma_r}) \left(d\mathrm{Vol}_{\Sigma_r} - d\mathrm{Vol}^\circ_{\Sigma_r} \right) + \mathcal{O}(\rho^4) \\
&= -\int_{\Sigma_r} \mathring{g}(J\mathring{\nabla} f^{(j)}, \vec{H}_{\Sigma_r} - \vec{H}^\circ_{\Sigma_r}) \, d\mathrm{Vol}^\circ_{\Sigma_r} \\
&\quad + \int_{\Sigma_r} \mathring{H}(\Sigma_r) \left(\mathring{\nabla} f^{(j)} \right) \left(d\mathrm{Vol}_{\Sigma_r} - d\mathrm{Vol}^\circ_{\Sigma_r} \right) + \mathcal{O}(\rho^4)
\end{aligned}
$$

since the integrals $\int_{\Sigma_r} \mathring{g}(J\mathring{\nabla} f^{(j)}, \vec{H}^\circ_{\Sigma_r}) \, d\mathrm{Vol}^\circ_\Sigma$ all vanish. The desired formula follows by combining this result with our earlier calculation. □

The following proposition now relates the projection map G_ρ to $\mathcal{G}_r$.

Proposition 15 *The mapping $G_\rho : U(M) \to \mathbb{R}^6 \approx \mathbb{C}^3$ satisfies*

$$
G_\rho(\mathcal{U}_p) = 4\pi^2 r_1 r_2 \, \rho^3 \mathcal{G}_r(\mathcal{U}_p) + \mathcal{O}(\rho^4)
$$

where

$$
\mathcal{G}_r(\mathcal{U}_p) := \mathrm{i} \begin{pmatrix} r_1^2 R^{\mathbb{C}}_{1\bar{1}1\bar{1};\bar{1}} + r_2^2 R^{\mathbb{C}}_{2\bar{2}2\bar{2};\bar{1}} \\ r_1^2 R^{\mathbb{C}}_{1\bar{1}1\bar{1};\bar{2}} + r_2^2 R^{\mathbb{C}}_{2\bar{2}2\bar{2};\bar{2}} \\ 2r_2^2 R^{\mathbb{C}}_{2\bar{2}2\bar{1}} - 2r_1^2 R^{\mathbb{C}}_{1\bar{1}2\bar{1}} \end{pmatrix} \in \mathbb{C}^3 \approx \mathbb{R}^6.
$$

Proof We expand the terms appearing in (17). After some preliminaries, we break up the proof into three sections corresponding to each of the three terms appearing there.

Let us assume that we have chosen local complex normal coordinates and performed the re-scaling (6). The ambient metric in complex notation is then given by

$$
g_\rho = 2\,\mathbf{Re}\left[\sum_{I,J}\left[F_\rho(z,\bar z)\right]_{,I\bar J}\,\mathrm{d}z^I \otimes \mathrm{d}\bar z^J\right]
$$

$$
= \mathbf{Re}\left[\sum_{I,J}\left(\delta_{IJ} + 2\rho^2\left[\hat F_\rho(z,\bar z)\right]_{,I\bar J}\right)\mathrm{d}z^I \otimes \mathrm{d}\bar z^J\right] \tag{18}
$$

where $F_\rho(z,\bar z) := \frac{1}{2}\|z\|^2 + \rho^2 \hat F_\rho(z,\bar z)$ is the Kähler potential. Our summations will be indexed by capital letters running from 1 to 2, and we will use a comma to denote partial differentiation with respect to z^I or $\hat z^J$. Let us write

$$
[\hat F_\rho(z,\bar z)]_{,I\bar J} := [F(z,\bar z)]^{(2)}_{,I\bar J} + \rho[F(z,\bar z)]^{(3)}_{,I\bar J} + \mathcal{O}(\rho^2)
$$

where $F^{(2)}$ and $F^{(3)}$ are terms in the Taylor expansion of the (un-scaled) Kähler potential F:

$$
[F(z,\bar z)]^{(2)}_{,I\bar J} = \frac{1}{2!}\sum_{L,M}\left(F_{,I\bar JLM}(0)z^L z^M + F_{,I\bar J\bar L\bar M}(0)\bar z^L \bar z^M + 2F_{,I\bar JL\bar M}(0)z^L \bar z^M\right)
$$

$$
[F(z,\bar z)]^{(3)}_{,I\bar J} = \frac{1}{3!}\sum_{L,M,N}\left(F_{,I\bar JLMN}(0)z^L z^M z^N + F_{,I\bar J\bar L\bar M\bar N}(0)\bar z^L \bar z^M \bar z^N\right.
$$
$$
\left. + 3F_{,I\bar JLM\bar N}(0)z^L z^M \bar z^N + 3F_{,I\bar JL\bar M\bar N}(0)z^L \bar z^M \bar z^N\right).
$$

To compress the notation somewhat in the ensuing calculations, we will replace "$F_{****}(0)$" and "$F_{*****}(0)$", which are the various fourth and fifth derivatives of F evaluated at zero, by "F_{****}" and "F_{*****}", respectively, in the Taylor expansions.

The difference of the volume forms calculation. The ambient metric h on the torus Σ_r is obtained by substituting $z = \left(r_1 e^{i\theta^1}, r_2 e^{i\theta^2}\right) := r e^{i\theta}$ into (18), yielding

$$
h = \sum_I r_I^2 (\mathrm{d}\theta^I)^2 + 2\rho^2 \sum_{I,J} r_I r_J \mathbf{Re}\left[e^{i(\theta^I - \theta^J)}\left[\hat F_\rho(re^{i\theta}, re^{-i\theta})\right]_{,I\bar J}\right]\mathrm{d}\theta^I \otimes \mathrm{d}\theta^J. \tag{19}
$$

The associated volume form is $\mathrm{dVol}_{\Sigma_r} := \sqrt{\det(h)}\,\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2$. Using the expansion $\sqrt{\det(A + \rho^2 B)} = \sqrt{\det(A)}\left(1 + \frac{1}{2}\rho^2 \mathrm{Tr}(A^{-1}B)\right) + \mathcal{O}(\rho^4)$, we obtain

$$
\mathrm{dVol}_{\Sigma_r} - \mathrm{dVol}^\circ_{\Sigma_r} = r_1 r_2\left(1 + \rho^2 \sum_I \mathbf{Re}\left[\hat F_\rho(re^{i\theta}, re^{-i\theta})\right]_{,I\bar I}\right)\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2 + \mathcal{O}(\rho^4) - \mathrm{dVol}^\circ_{\Sigma_r}
$$

$$
= r_1 r_2\left(\rho^2 \sum_I \mathbf{Re}\left[F(re^{i\theta}, re^{-i\theta})\right]^{(2)}_{,I\bar I} + \rho^3 \sum_I \mathbf{Re}\left[F(re^{i\theta}, re^{-i\theta})\right]^{(3)}_{,I\bar I}\right)\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2 + \mathcal{O}(\rho^4)
$$

since only the quadratic and cubic terms in the series expansion of F_ρ in ρ contribute to the ρ^2 and ρ^3 terms in the expansion of $\sqrt{\det(h)}$.

We must now multiply the preceding by $\mathring H(\mathring\nabla f^{(j)})$ for $j = 1, \ldots, 6$, and integrate over the torus. There are two cases to consider: $j = 1, \ldots, 4$, corresponding to the co-kernel generators coming from translation; and $j = 5, 6$, corresponding to the co-kernel generators corresponding to $U(2)$-rotation.

Case 1. In this case $f^{(j)}$ is one of either $r_s \cos(\theta^s)$ or $r_s \sin(\theta^s)$ for $s = 1, 2$. It is thus the case that the integrals of $\mathring{H}(\mathring{\nabla} f^{(j)}) \cdot \mathbf{Re}\left[F(re^{i\theta}, re^{-i\theta})\right]^{(2)}_{,I\bar{I}}$ over the torus must all vanish. Furthermore we have $\mathring{H}(\mathring{\nabla} f^{(j)}) = -(r_1^{-2} f^{(j)}_{,1} + r_2^{-2} f^{(j)}_{,2})$ and thus

$$-\mathring{H}(\mathring{\nabla} f^{(j)}) = \begin{cases} -r_1^{-1} \sin(\theta^1) & j = 1 \\ r_1^{-1} \cos(\theta^1) & j = 2 \\ -r_2^{-1} \sin(\theta^2) & j = 3 \\ r_2^{-1} \cos(\theta^2) & j = 4. \end{cases}$$

Hence we are left with computing the integrals

$$I_0^{(j)} := -w_2^{(j)} \int_{\Sigma_r} \mathring{H}(\mathring{\nabla} f^{(j)}) \left(\mathrm{dVol}_{\Sigma_r} - \mathrm{dVol}^{\circ}_{\Sigma_r}\right)$$

$$= -r_1 r_2 \rho^3 \sum_I \int_{\mathbb{T}^2} \mathring{H}(\mathring{\nabla} f^{(j)}) \,\mathbf{Re}\left[F(re^{i\theta}, re^{-i\theta})\right]^{(3)}_{,I\bar{I}} \, d\theta^1 \wedge d\theta^2 + \mathcal{O}(\rho^4)$$

$$= -\frac{r_1 r_2 \rho^3}{6} \sum_{I,K,L,M} \int_{\mathbb{T}^2} r_K r_L r_M \mathring{H}(\mathring{\nabla} f^{(j)}) \mathbf{Re}\left[F_{,I\bar{I}KLM}\, e^{i(\theta^K + \theta^L + \theta^M)} \right.$$

$$+ 3 F_{,I\bar{I}KL\bar{M}}\, e^{i(\theta^K + \theta^L - \theta^M)} + 3 F_{,I\bar{I}K\bar{L}\bar{M}}\, e^{i(\theta^K - \theta^L - \theta^M)}$$

$$\left. + F_{,I\bar{I}\bar{K}\bar{L}\bar{M}}\, e^{-i(\theta^K + \theta^L + \theta^M)} \right] d\theta^1 \wedge d\theta^2 + \mathcal{O}(\rho^4).$$

Now in order to proceed, it is most convenient to calculate these integrals two-by-two:

$$I_0^{(2S-1)} + i\, I_0^{(2S)}$$

$$= \frac{r_1 r_2 \rho^3}{6 r_S} \sum_{I,K,L,M} \int_{\mathbb{T}^2} r_K r_L r_M i e^{i\theta^S} \mathbf{Re}\left[F_{,I\bar{I}KLM}\, e^{i(\theta^K + \theta^L + \theta^M)} + 3 F_{,I\bar{I}KL\bar{M}}\, e^{i(\theta^K + \theta^L - \theta^M)} \right.$$

$$\left. + 3 F_{,I\bar{I}K\bar{L}\bar{M}}\, e^{i(\theta^K - \theta^L - \theta^M)} + F_{,I\bar{I}\bar{K}\bar{L}\bar{M}}\, e^{-i(\theta^K + \theta^L + \theta^M)} \right] d\theta^1 \wedge d\theta^2 + \mathcal{O}(\rho^4)$$

$$= \frac{i r_1 r_2 \rho^3}{2 r_S} \sum_{I,K,L,M} \int_{\mathbb{T}^2} r_K r_L r_M F_{,I\bar{I}K\bar{L}\bar{M}}\, e^{i(\theta^S + \theta^K - \theta^L - \theta^M)} \, d\theta^1 \wedge d\theta^2 + \mathcal{O}(\rho^4),$$

since the sum over K, L, M of the terms in the brackets is already real, and since the only terms in this expansion that will survive the integration over the torus are those for which it is possible to arrange $\theta^S + \theta^K \pm \theta^L \pm \theta^M = 0$, in which case the value of the integral is $4\pi^2$. These are clearly the $(\theta^S + \theta^K - \theta^L - \theta^M)$-terms. Amongst these, the ones yielding non-vanishing integrals are given in the following table

S	K	L	M
1	1	1	1
1	2	1	2
1	2	2	1
2	1	1	2
2	1	2	1
2	2	2	2

with the resulting expansions

$$I_0^{(2S-1)} + i\, I_0^{(2S)} = \begin{cases} 2i\pi^2 r_1 r_2 \rho^3 \sum_I \left(r_1^2 F_{,I\bar{1}1\bar{1}\bar{1}} + 2r_2^2 F_{,I\bar{1}2\bar{2}\bar{1}}\right) + \mathcal{O}(\rho^4) & S = 1 \\ 2i\pi^2 r_1 r_2 \rho^3 \sum_I \left(2r_1^2 F_{,I\bar{1}1\bar{1}\bar{2}} + r_2^2 F_{,I\bar{1}2\bar{2}\bar{2}}\right) + \mathcal{O}(\rho^4) & S = 2. \end{cases}$$

Case 2. In this case $f^{(j)}$ is one of either $r_1 r_2 \cos(\theta^1 - \theta^2)$ or $r_1 r_2 \sin(\theta^1 - \theta^2)$. It is thus the case that the integrals of $\mathring{H}(\mathring{\nabla} f^{(j)}) \cdot \mathbf{Re}\left[F(re^{i\theta}, re^{-i\theta})\right]_{,I\bar{I}}^{(3)}$ over the torus must all vanish. Furthermore

$$-\mathring{H}(\mathring{\nabla} f^{(j)}) = \begin{cases} -r_1 r_2 \left(r_1^{-2} - r_2^{-2}\right) \sin(\theta^1 - \theta^2) & j = 5 \\ r_1 r_2 \left(r_1^{-2} - r_2^{-2}\right) \cos(\theta^1 - \theta^2) & j = 6. \end{cases}$$

Hence we are left with computing the integrals

$$\begin{aligned}
I_0^{(j)} :&= -w_2^{(j)} \int_{\Sigma_r} \mathring{H}(\mathring{\nabla} f^{(j)}) \left(\mathrm{dVol}_{\Sigma_r} - \mathrm{dVol}_{\Sigma_r}^{\circ}\right) \\
&= -r_1 r_2 \rho^3 \sum_I \int_{\mathbb{T}^2} \mathring{H}(\mathring{\nabla} f^{(j)}) \,\mathbf{Re}\left[F(re^{i\theta}, re^{-i\theta})\right]_{,I\bar{I}}^{(2)} \, d\theta^1 \wedge d\theta^2 + \mathcal{O}(\rho^4) \\
&= -\frac{r_1 r_2 \rho^3}{2} \sum_{I,K,L} \int_{\mathbb{T}^2} r_K r_L \mathring{H}(\mathring{\nabla} f^{(j)}) \mathbf{Re}\left[F_{,I\bar{I}KL} e^{i(\theta^K + \theta^L)}\right. \\
&\qquad\left. + 2F_{,I\bar{I}K\bar{L}} e^{i(\theta^K - \theta^L)} + F_{,I\bar{I}\bar{K}\bar{L}} e^{-i(\theta^K + \theta^L)}\right] d\theta^1 \wedge d\theta^2 + \mathcal{O}(\rho^4).
\end{aligned}$$

As above, it is most convenient to calculate these integrals two-by-two:

$$\begin{aligned}
I_0^{(5)} + i\, I_0^{(6)} &= \frac{r_1^2 r_2^2 \left(r_1^{-2} - r_2^{-2}\right) \rho^3}{2} \sum_{I,K,L} \int_{\mathbb{T}^2} r_K r_L i e^{i(\theta^1 - \theta^2)} \mathbf{Re}\left[F_{,I\bar{I}KL} e^{i(\theta^K + \theta^L)}\right. \\
&\qquad\left. + 2F_{,I\bar{I}K\bar{L}} e^{i(\theta^K - \theta^L)} + F_{,I\bar{I}\bar{K}\bar{L}} e^{-i(\theta^K + \theta^L)}\right] d\theta^1 \wedge d\theta^2 + \mathcal{O}(\rho^4) \\
&= i r_1^2 r_2^2 \left(r_1^{-2} - r_2^{-2}\right) \rho^3 \sum_{I,K,L} \int_{\mathbb{T}^2} r_K r_L F_{,I\bar{I}K\bar{L}} e^{i(\theta^1 - \theta^2 + \theta^K - \theta^L)} d\theta^1 \wedge d\theta^2 + \mathcal{O}(\rho^4)
\end{aligned}$$

since the sum of the terms in the brackets over K, L is already real, and the only terms in this expansion that will survive the integration over the torus are those for which it is possible to arrange $\theta^1 - \theta^2 \pm \theta^K \pm \theta^L = 0$. These are clearly the $(\theta^1 - \theta^2 + \theta^K - \theta^L)$-terms. Amongst these, the ones yielding non-vanishing integrals are $K = 2, L = 1$, with the resulting expansion

$$I_0^{(5)} + i\, I_0^{(6)} = 4i\pi^2 r_1 r_2 \left(r_2^2 - r_1^2\right) \rho^3 \sum_I F_{,I\bar{I}2\bar{1}} + \mathcal{O}(\rho^4).$$

The difference of the mean curvatures calculation. Recall that the tangent vectors of Σ_r are given in complex notation by $E_K := i r_K e^{i\theta^K} \frac{\partial}{\partial z^K}$, for $K = 1, 2$. Since the component $i z^K$ of E_K is holomorphic, we have $D_{E_K} \bar{E}_L = 0$, and so the covariant derivatives of the tangent vector fields with respect to g are given by, in complex notation,

$$D_{E_K} E_L = \frac{\partial}{\partial \theta^K}(ir_L e^{i\theta^L}) \frac{\partial}{\partial z^L} - r_K r_L e^{i(\theta^K + \theta^L)} D_{\frac{\partial}{\partial z^K}} \frac{\partial}{\partial z^L}$$

$$= \delta_{KL} J E_L - r_K r_L e^{i(\theta^K + \theta^L)} \sum_M \Gamma^M_{KL} \frac{\partial}{\partial z^M}.$$

We have used the fact that $\Gamma^{\bar{M}}_{KL} = 0$. Now to continue, we deduce from the expansion (18) that

$$\Gamma^M_{KL} = \sum_J g^{M\bar{J}}_\rho \frac{\partial g_{K\bar{J}}}{\partial z^L} = 2\rho^2 \sum_J \delta^{M\bar{J}} \frac{\partial}{\partial z^L} \left[\hat{F}_\rho(z, \bar{z}) \right]_{,K\bar{J}} + \mathcal{O}(\rho^4)$$

$$= 2\rho^2 \left[\hat{F}_\rho(z, \bar{z}) \right]_{,LK\bar{M}} + \mathcal{O}(\rho^4).$$

Thus we conclude

$$D_{E_K} E_L = \delta_{KL} J E_L - 2r_K r_L e^{i(\theta^K + \theta^L)} \rho^2 \sum_M \left[\hat{F}_\rho(z, \bar{z}) \right]_{,LK\bar{M}} \frac{\partial}{\partial z^M} + \mathcal{O}(\rho^4).$$

The next step is to take the trace of the quantity above with respect to the induced metric h of Σ_r. Using the components of h given in Eq. 19, we deduce

$$h^{KL} = \frac{1}{r_K r_L} \left(\delta^{KL} - 2\rho^2 \mathbf{Re} \left(e^{i(\theta^K - \theta^L)} \left[\hat{F}_\rho(z, \bar{z}) \right]_{,K\bar{L}} \right) \right) + \mathcal{O}(\rho^4),$$

and hence,

$$\sum_{K,L} h^{KL} D_{E_K} E_L = \vec{H}^\circ_{\Sigma_r} - \sum_K \frac{2\rho^2}{r_K^2} \mathbf{Re} \left(\left[\hat{F}_\rho(z, \bar{z}) \right]_{,K\bar{K}} \right) J E_K$$

$$- 2\rho^2 \sum_{K,M} e^{2i\theta^K} \left[\hat{F}_\rho(z, \bar{z}) \right]_{,KK\bar{M}} \frac{\partial}{\partial z^M} + \mathcal{O}(\rho^4).$$

For the next step in the calculation, we note that the tangential gradient of $f^{(j)}$ is equal to

$$\mathring{\nabla} f^{(j)} = \sum_S \frac{1}{r_S^2} \frac{\partial f^{(j)}}{\partial \theta^S} E_S$$

so that, using the fact that $\vec{H}_{\Sigma_r} - \sum_{K,L} h^{KL} D_{E_K} E_L$ is tangent to Σ_r, we get

$$\mathring{g}(J\mathring{\nabla} f^{(j)}, \vec{H}_{\Sigma_r} - \vec{H}^\circ_{\Sigma_r}) = \mathring{g}\left(\sum_S \frac{1}{r_S^2} \frac{\partial f^{(j)}}{\partial \theta^S} J E_S, \sum_{K,L} h^{KL} D_{E_K} E_L - \vec{H}^\circ_{\Sigma_r} \right)$$

$$= (\mathrm{I}) + (\mathrm{II}) + \mathcal{O}(\rho^4),$$

where

$$(\mathrm{I}) = -2\rho^2 \sum_K \frac{1}{r_K^2} \left[\hat{F}_\rho(z, \bar{z}) \right]_{,K\bar{K}} \frac{\partial f^{(j)}}{\partial \theta^K} \tag{20a}$$

$$(\mathrm{II}) = -2\rho^2 \, \mathbf{Re} \sum_A dz^A \otimes d\bar{z}^A \left(\sum_{K,M} e^{2i\theta^K} \left[\hat{F}_\rho(z, \bar{z}) \right]_{,KK\bar{M}} \frac{\partial}{\partial z^M}, -\sum_S \frac{1}{r_S} e^{-i\theta^S} \frac{\partial f^{(j)}}{\partial \theta^S} \frac{\partial}{\partial \bar{z}^S} \right)$$

$$= 2\rho^2 \sum_{A,K} \frac{1}{r_A} \mathbf{Re} \left(e^{i(2\theta^K - \theta^A)} \left[\hat{F}_\rho(z, \bar{z}) \right]_{,KK\bar{A}} \right) \frac{\partial f^{(j)}}{\partial \theta^A}. \tag{20b}$$

We now turn to computing the expansions of the integrals corresponding to the terms above. Let $w_2^{(j)} \int_{\Sigma_r} \mathring{g}(J\mathring{\nabla} f^{(j)}, \vec{H}_{\Sigma_r} - \vec{H}_{\Sigma_r}^{\circ})\,\mathrm{dVol}_{\Sigma_r}^{\circ} = I_1^{(j)} + I_2^{(j)} + \mathcal{O}(\rho^4)$, where $I_1^{(j)}$ and $I_2^{(j)}$ correspond to expanding each of the two leading terms in (20):

$$I_1^{(j)} := -2\rho^2 w_2^{(j)} \int_{\Sigma_r} \sum_K \frac{1}{r_K^2} \left[\hat{F}_\rho(z,\bar{z})\right]_{,K\bar{K}} \frac{\partial f^{(j)}}{\partial \theta^K}\,\mathrm{dVol}_{\Sigma_r}^{\circ}$$

$$I_2^{(j)} := 2\rho^2 w_2^{(j)} \int_{\Sigma_r} \sum_{A,K} \frac{1}{r_A}\,\mathbf{Re}\left(e^{i(2\theta^K - \theta^A)}\left[\hat{F}_\rho(z,\bar{z})\right]_{,K K \bar{A}}\right) \frac{\partial f^{(j)}}{\partial \theta^A}\,\mathrm{dVol}_{\Sigma_r}^{\circ}.$$

We handle each term in two cases.

Case 1 For $S = 1, 2$ we write

$$I_1^{(2S-1)} + iI_1^{(2S)} = -2\rho^2 \int_{\Sigma_r} \sum_K \frac{1}{r_K^2} \left(\frac{\partial f^{(2S-1)}}{\partial \theta^K} + i\frac{\partial f^{(2S)}}{\partial \theta^K}\right)\left[\hat{F}_\rho(z,\bar{z})\right]_{,K\bar{K}}\,\mathrm{dVol}_{\Sigma_r}^{\circ}$$

$$= -2r_1 r_2 \rho^2 \int_{\mathbb{T}^2} \frac{ie^{i\theta^S}}{r_S}\left[\hat{F}_\rho(z,\bar{z})\right]_{,S\bar{S}}\,\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2.$$

Up to the error term, the only term in the Taylor expansion of $\left[\hat{F}_\rho(z,\bar{z})\right]_{,S\bar{S}}$ that contributes to the integral is the third-order term. Upon substituting $(z^1, z^2) = (r_1 e^{i\theta^1}, r_2 e^{i\theta^2})$ and arguing as before, we see that only one term in the Taylor expansion of $\left[\hat{F}_\rho(z,\bar{z})\right]_{,S\bar{S}}$ has the correct combination of $\pm\theta$ terms for a contribution to be possible. So we get

$$I_1^{(2S-1)} + iI_1^{(2S)} = -\frac{ir_1 r_2 \rho^3}{3rs} \sum_{L,M,N} \int_{\mathbb{T}^2} e^{i\theta^S} r_L r_M r_N \left(F_{,S\bar{S}LMN}e^{i(\theta^L + \theta^M + \theta^N)}\right.$$

$$+ 3F_{,S\bar{S}LM\bar{N}}e^{i(\theta^L + \theta^M - \theta^N)} + 3F_{,S\bar{S}L\bar{M}\bar{N}}e^{i(\theta^L - \theta^M - \theta^N)}$$

$$\left. + F_{,S\bar{S}\bar{L}\bar{M}\bar{N}}e^{-i(\theta^L + \theta^M + \theta^N)}\right)\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2 + \mathcal{O}(\rho^4)$$

$$= -\frac{ir_1 r_2 \rho^3}{rs} \sum_{L,M,N} r_L r_M r_N F_{,S\bar{S}L\bar{M}\bar{N}} \int_{\mathbb{T}^2} e^{i(\theta^S + \theta^L - \theta^M - \theta^N)}\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2 + \mathcal{O}(\rho^4)$$

$$= \begin{cases} -4i\pi^2 r_1 r_2 \rho^3 \left(r_1^2 F_{,1\bar{1}1\bar{1}\bar{1}} + 2r_2^2 F_{,1\bar{1}2\bar{2}\bar{1}}\right) + \mathcal{O}(\rho^4) & S = 1 \\ -4i\pi^2 r_1 r_2 \rho^3 \left(r_2^2 F_{,2\bar{2}2\bar{2}\bar{2}} + 2r_1^2 F_{,2\bar{2}1\bar{1}\bar{2}}\right) + \mathcal{O}(\rho^4) & S = 2. \end{cases}$$

Case 2 For $j = 5$ or $j = 6$, we note

$$\sum_K \frac{1}{r_K^2}\left(\frac{\partial f^{(5)}}{\partial \theta^K} + i\frac{\partial f^{(6)}}{\partial \theta^K}\right)\left[\hat{F}_\rho(z,\bar{z})\right]_{,K\bar{K}} = r_1 r_2 i e^{i(\theta^1 - \theta^2)}\left[\frac{1}{r_1^2}\left[\hat{F}_\rho(z,\bar{z})\right]_{,1\bar{1}} - \frac{1}{r_2^2}\left[\hat{F}_\rho(z,\bar{z})\right]_{,2\bar{2}}\right].$$

Note that up to the $\mathcal{O}(\rho^4)$ error term, the only term in the Taylor expansion of $\left[\hat{F}_\rho(z,\bar{z})\right]_{,K\bar{K}}$ that contributes to the integral this time is the second-order term. Upon making the substitution $(z^1, z^2) = (r_1 e^{i\theta^1}, r_2 e^{i\theta^2})$ and arguing as before, we see also that only one term in this expansion has the correct combination of $\pm\theta$ terms for a contribution to be possible. Thus we obtain

$$
I_1^{(5)} + iI_1^{(6)} = -ir_1^2 r_2^2 \rho^3 \sum_{L,M} r_L r_M
$$

$$
\times \int_{\mathbb{T}^2} \left[\frac{e^{i(\theta^1 - \theta^2)}}{r_1^2} \left(F_{,1\bar{1}LM} e^{i(\theta^L + \theta^M)} + 2F_{,1\bar{1}L\bar{M}} e^{i(\theta^L - \theta^M)} + F_{,1\bar{1}\bar{L}\bar{M}} e^{-i(\theta^L + \theta^M)} \right) \right.
$$

$$
\left. - \frac{e^{i(\theta^1 - \theta^2)}}{r_2^2} \left(F_{,2\bar{2}LM} e^{i(\theta^L + \theta^M)} + 2F_{,2\bar{2}L\bar{M}} e^{i(\theta^L - \theta^M)} + F_{,2\bar{2}\bar{L}\bar{M}} e^{-i(\theta^L + \theta^M)} \right) \right] d\theta^1 \wedge d\theta^2
$$

$$
= 8i\pi^2 r_1 r_2 \rho^3 \left(r_1^2 F_{,2\bar{2}2\bar{1}} - r_2^2 F_{,1\bar{1}2\bar{1}} \right) + \mathcal{O}(\rho^4).
$$

We now perform the corresponding calculation for the term (II) in (20). We first note the Taylor expansion

$$
\left[\hat{F}_\rho(z, \bar{z}) \right]_{,KK\bar{A}} = \sum_L \left(F_{,KK\bar{A}L} z^L + F_{,KK\bar{A}\bar{L}} \bar{z}^L \right)
$$

$$
+ \frac{\rho}{2} \sum_{L,M} \left(F_{,KK\bar{A}LM} z^L z^M + 2F_{,KK\bar{A}L\bar{M}} z^L \bar{z}^M \right.
$$

$$
\left. + F_{,KK\bar{A}\bar{L}\bar{M}} \bar{z}^L \bar{z}^M \right) + \mathcal{O}(\rho^2).
$$

Case 1 As above, for $S = 1, 2$ we write

$$
I_2^{(2S-1)} + iI_2^{(2S)} := 2\rho^2 \int_{\Sigma_r} \sum_{A,K} \left(\frac{\partial f^{(2S-1)}}{\partial \theta^A} + i \frac{\partial f^{(2S)}}{\partial \theta^A} \right) \mathbf{Re} \left[r_A^{-1} e^{i(2\theta^K - \theta^A)} \left[\hat{F}_\rho(z, \bar{z}) \right]_{,KK\bar{A}} \right]
$$

$$
\times d\mathrm{Vol}^\circ_{\Sigma_r}
$$

$$
= 2\rho^2 \int_{\Sigma_r} ie^{i\theta^S} \mathbf{Re} \left[\sum_K e^{i(2\theta^K - \theta^S)} \left[\hat{F}_\rho(z, \bar{z}) \right]_{,KK\bar{S}} \right] d\mathrm{Vol}^\circ_{\Sigma_r}
$$

$$
= ir_1 r_2 \rho^3 \sum_{K,L,M} r_L r_M \int_{\mathbb{T}^2} e^{i\theta^S} \mathbf{Re} \left[e^{i(2\theta^K - \theta^S)} \left(F_{,KK\bar{S}LM} e^{i(\theta^L + \theta^M)} \right. \right.
$$

$$
\left. \left. + 2F_{,KK\bar{S}L\bar{M}} e^{i(\theta^L - \theta^M)} + F_{,KK\bar{S}\bar{L}\bar{M}} e^{-i(\theta^L + \theta^M)} \right) \right] d\theta^1 \wedge d\theta^2 + \mathcal{O}(\rho^4).
$$

We can re-write the integrand as follows:

$$
e^{i\theta^S} \mathbf{Re} \left[e^{i(2\theta^K - \theta^S)} \left(F_{,KK\bar{S}LM} e^{i(\theta^L + \theta^M)} + 2F_{,KK\bar{S}L\bar{M}} e^{i(\theta^L - \theta^M)} + F_{,KK\bar{S}\bar{L}\bar{M}} e^{-i(\theta^L + \theta^M)} \right) \right]
$$

$$
= \frac{1}{2} e^{2i\theta^K} \left(F_{,KK\bar{S}LM} e^{i(\theta^L + \theta^M)} + 2F_{,KK\bar{S}L\bar{M}} e^{i(\theta^L - \theta^M)} + F_{,KK\bar{S}\bar{L}\bar{M}} e^{-i(\theta^L + \theta^M)} \right)
$$

$$
+ \frac{1}{2} e^{i(-2\theta^K + 2\theta^S)} \left(F_{,\bar{K}\bar{K}S\bar{L}\bar{M}} e^{-i(\theta^L + \theta^M)} + 2F_{,\bar{K}\bar{K}S\bar{L}\bar{M}} e^{i(-\theta^L + \theta^M)} + F_{,\bar{K}\bar{K}SLM} e^{i(\theta^L + \theta^M)} \right).
$$

We can now easily integrate, and note that most terms disappear upon integration over the torus:

$$
\begin{aligned}
I_2^{(2S-1)} + iI_2^{(2S)} &= \frac{ir_1 r_2 \rho^3}{2} \sum_{K,L,M} r_L r_M \int_{\mathbb{T}^2} \left(e^{i(2\theta^K - \theta^L - \theta^M)} F_{,KK\bar{S}\bar{L}\bar{M}} \right. \\
&\qquad \left. + 2e^{i(-2\theta^K + 2\theta^S - \theta^L + \theta^M)} F_{,\bar{K}\bar{K}S\bar{L}M} \right) d\theta^1 \wedge d\theta^2 + \mathcal{O}(\rho^4) \\
&= 2i\pi^2 r_1 r_2 \rho^3 \sum_L r_L^2 \left(F_{,L\bar{L}L\bar{L}\bar{S}} + 2F_{,S\bar{S}L\bar{L}\bar{S}} \right) + \mathcal{O}(\rho^4) \\
&= \begin{cases} 2i\pi^2 r_1 r_2 \rho^3 \left(3r_1^2 F_{,1\bar{1}1\bar{1}\bar{1}} + r_2^2 F_{,2\bar{2}2\bar{2}\bar{1}} + 2r_2^2 F_{,1\bar{1}2\bar{2}\bar{1}} \right) + \mathcal{O}(\rho^4) & S = 1 \\ 2i\pi^2 r_1 r_2 \rho^3 \left(r_1^2 F_{,1\bar{1}1\bar{1}\bar{2}} + 2r_1^2 F_{,2\bar{2}1\bar{1}\bar{2}} + 3r_2^2 F_{,2\bar{2}2\bar{2}\bar{2}} \right) + \mathcal{O}(\rho^4) & S = 2. \end{cases}
\end{aligned}
$$

Case 2 Finally, we can compute

$$
\begin{aligned}
I_2^{(5)} + iI_2^{(6)} &= 2\rho^3 \int_{\Sigma_r} \sum_{A,K} \left(\frac{\partial f^{(5)}}{\partial \theta^A} + i\frac{\partial f^{(6)}}{\partial \theta^A} \right) \mathbf{Re} \left[r_A^{-1} e^{i(2\theta^K - \theta^A)} \left[\hat{F}_\rho(z, \bar{z}) \right]_{,KK\bar{A}} \right] d\text{Vol}_{\Sigma_r}^\circ \\
&= 2i(r_1 r_2)^2 \rho^3 \int_{\mathbb{T}^2} e^{i(\theta^1 - \theta^2)} \mathbf{Re} \left[\sum_{A,K} (-1)^{A+1} r_A^{-1} e^{i(2\theta^K - \theta^A)} \left[\hat{F}_\rho(z, \bar{z}) \right]_{,KK\bar{A}} \right] d\theta^1 \wedge d\theta^2.
\end{aligned}
$$

We expand the integrand, and note that up to $\mathcal{O}(\rho^4)$, only the linear terms in the expansion will contribute. These terms in the integrand are precisely

$$
\begin{aligned}
\frac{e^{i(\theta^1 - \theta^2)}}{2} &\left[\sum_{A,K} (-1)^{A+1} r_A^{-1} e^{i(2\theta^K - \theta^A)} \sum_L r_L \left(F_{,KK\bar{A}\bar{L}} e^{i\theta^L} + F_{,KK\bar{A}\bar{L}} e^{-i\theta^L} \right) \right. \\
&\left. + \sum_{A,K} (-1)^{A+1} r_A^{-1} e^{i(-2\theta^K + \theta^A)} \sum_L r_L \left(F_{,\bar{K}\bar{K}A\bar{L}} e^{-i\theta^L} + F_{,\bar{K}\bar{K}AL} e^{i\theta^L} \right) \right].
\end{aligned}
$$

There are only four of these terms which survive integration: on the first line above these correspond to $(K, A, L) = (2, 1, 2)$ or $(2, 2, 1)$, and from the second line above to $(K, A, L) = (1, 1, 2)$ or $(1, 2, 1)$. Thus upon integration we get

$$
\begin{aligned}
I_2^{(5)} + iI_2^{(6)} &= 4\pi^2 i(r_1 r_2)^2 \rho^3 \left[F_{,2\bar{2}1\bar{2}}(r_1^{-1} r_2 - r_2^{-1} r_1) + F_{,\bar{1}\bar{1}12}(r_1^{-1} r_2 - r_2^{-1} r_1) \right] + \mathcal{O}(\rho^4) \\
&= 4i\pi^2 r_1 r_2 (r_2^2 - r_1^2) \rho^3 \left(F_{,2\bar{2}2\bar{1}} + F_{,1\bar{1}2\bar{1}} \right) + \mathcal{O}(\rho^4).
\end{aligned}
$$

The symplectic form calculation. The ambient symplectic form, expressed in the re-scaled complex coordinates, is given by

$$
\omega_\rho = \mathring{\omega} - 2\rho^2 \mathbf{Im} \left[\sum_{I,J} \left[\hat{F}_\rho(z, \bar{z}) \right]_{,I\bar{J}} dz^I \otimes d\bar{z}^J \right]
$$

and on Σ_r where $\mathring{\omega}$ vanishes, this becomes

$$
\omega_\rho - \mathring{\omega} = -2r_1 r_2 \rho^2 \mathbf{Im} \left[\left[\hat{F}_\rho(re^{i\theta}, re^{-i\theta}) \right]_{,1\bar{2}} e^{i(\theta^1 - \theta^2)} \right] d\theta^1 \wedge d\theta^2.
$$

We'll once again expand $\left[F_\rho(z,\bar{z})\right]_{I\bar{J}}$ using the Taylor expansion of the Kähler potential. We find (where "c.c." means the complex conjugate of the preceding term inside the brackets)

$$\omega_\rho - \mathring{\omega}$$

$$= \frac{\mathrm{i}r_1 r_2 \rho^2}{2} \sum_{K,L}\left[\left(F_{,1\bar{2}KL}\,z^K z^L + 2F_{,1\bar{2}K\bar{L}}\,z^K \bar{z}^L + F_{,1\bar{2}\bar{K}\bar{L}}\,\bar{z}^K \bar{z}^L\right)\mathrm{e}^{\mathrm{i}(\theta^1 - \theta^2)} - \text{c.c.}\right]\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2$$

$$+ \frac{\mathrm{i}r_1 r_2 \rho^3}{6} \sum_{K,L,M}\left[\left(\begin{matrix}F_{,1\bar{2}KLM}\,z^K z^L z^M + 3F_{,1\bar{2}KL\bar{M}}\,z^K z^L \bar{z}^M \\ +3F_{,1\bar{2}K\bar{L}\bar{M}}\,z^K \bar{z}^L \bar{z}^M + F_{,1\bar{2}\bar{K}\bar{L}\bar{M}}\,\bar{z}^K \bar{z}^L \bar{z}^M\end{matrix}\right)\mathrm{e}^{\mathrm{i}(\theta^1 - \theta^2)} - \text{c.c.}\right]\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2 + \mathcal{O}(\rho^4).$$

Now, if we set $I_\omega^{(s)} := w_1^{(s)} \int_{\Sigma_r} f^{(s)}(\omega_\rho - \mathring{\omega})$ for $s = 1, \dots, 4$, and compute these integrals two-by-two in the same way as above, we find for $S = 1, 2$,

$$I_\omega^{(2S-1)} + \mathrm{i}I_\omega^{(2S)} = \frac{(-1)^{S+1}}{r_1 r_2} \int_{\Sigma_r} z^S(\omega_\rho - \mathring{\omega}).$$

Recall also that our co-kernel calculations show that the $U(2)$-rotations do not contribute in the $\omega_\rho - \mathring{\omega}$ integrals, i.e. $I_\omega^{(5)} = 0 = I_\omega^{(6)}$. Now, in computing these integrals, it is clear that only terms having even power in z or $\bar{z}$ survive the integration over Σ_r. Moreover, only those terms having the correct combination of $\pm\theta$ terms will survive the integration. Thus we find

$$I_\omega^{(2S-1)} + \mathrm{i}I_\omega^{(2S)}$$

$$= \sum_{K,L,M} \frac{\mathrm{i}(-1)^{S+1}\rho^3 r_S r_K r_L r_M}{2}\left(F_{,1\bar{2}K\bar{L}M}\int_{\mathbb{T}^2}\mathrm{e}^{\mathrm{i}(\theta^S+\theta^1-\theta^2+\theta^K-\theta^L-\theta^M)}\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2\right.$$

$$\left.- F_{,\bar{1}2\bar{K}L\bar{M}}\int_{\mathbb{T}^2}\mathrm{e}^{\mathrm{i}(\theta^S-\theta^1+\theta^2-\theta^K-\theta^L+\theta^M)}\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2\right) + \mathcal{O}(\rho^5).$$

The following values of S, K, L, M yield non-zero integrals.

First integral:	S	K	L	M	Second integral:	S	K	L	M
	1	2	1	1		1	1	2	1
	2	1	1	1		1	2	1	1
	2	2	1	2		1	2	2	2
	2	2	2	1		2	2	2	1

Therefore

$$I_\omega^{(2S-1)} + \mathrm{i}I_\omega^{(2S)} = \begin{cases} -2\mathrm{i}\pi^2 r_1 r_2 \rho^3 \left(r_1^2 F_{,1\bar{2}2\bar{1}\bar{1}} + r_2^2 F_{,1\bar{2}2\bar{2}\bar{2}}\right) + \mathcal{O}(\rho^5) & S = 1 \\ -2\mathrm{i}\pi^2 r_1 r_2 \rho^3 \left(r_1^2 F_{,\bar{2}1\bar{1}1\bar{1}} + r_2^2 F_{,2\bar{1}1\bar{2}\bar{2}}\right) + \mathcal{O}(\rho^5) & S = 2. \end{cases}$$

To complete the proof of Proposition 15 we just combine all the pieces computed above:

$$I_\rho^{(2S-1)} + \mathrm{i}I_\rho^{(2S)} = \left(I_0^{(2S-1)} + \mathrm{i}I_0^{(2S)}\right) + \left(I_1^{(2S-1)} + \mathrm{i}I_1^{(2S)}\right) + \left(I_2^{(2S-1)} + \mathrm{i}I_2^{(2S)}\right)$$

$$+ \left(I_\omega^{(2S-1)} + \mathrm{i}I_\omega^{(2S)}\right)$$

for $S = 1, 2, 3$. To conclude, we use the fact that $F_{,J\bar{K}L\bar{M}} = R^{\mathbb{C}}_{J\bar{K}L\bar{M}}(p)$. $\qquad\square$

5.5 The proof of the main theorem

In this section, we conclude the proof of the Main Theorem by showing that if $\mathcal{G}_r$ has a $\boldsymbol{\Delta}$-non-degenerate zero at $\mathcal{U}_p \in \boldsymbol{U}(M)$, i.e. if $[\mathcal{U}_p]$ is a non-degenerate critical point of $\mathcal{F}_r$, then it is possible to find a nearby frame $\mathcal{U}_{p'}$ for which $G_\rho(\mathcal{U}_{p'}) = 0$. In this case $\tilde{\Sigma}_r(\mathcal{U}_{p'})$ is an exactly Hamiltonian stationary Lagrangian submanifold. This will then complete the proof of the Main Theorem.

Theorem 16 *Suppose $\mathcal{U}_p$ is a $\boldsymbol{\Delta}$-non-degenerate zero of $\mathcal{G}_r$. If ρ is sufficiently small, then there is $\mathcal{U}_{p'}$ near $\mathcal{U}_p$ so that the submanifold $\tilde{\Sigma}_r(\mathcal{U}_{p'})$ that was obtained via Theorem 13 from the torus $\Sigma_r(\mathcal{U}_{p'})$ is a Hamiltonian stationary Lagrangian submanifold. The distance between $\mathcal{U}_p$ and $\mathcal{U}_{p'}$ is $\mathcal{O}(\rho)$.*

Proof We must to find $\mathcal{U}_{p'}$ so that $G_\rho(\mathcal{U}_{p'})$ vanishes. The estimate of Proposition 15 says

$$G_\rho(\mathcal{U}_p) = 4\pi^2 r_1 r_2 \, \rho^3 \, \mathcal{G}_r(\mathcal{U}_p) + \mathcal{O}(\rho^4).$$

Suppose now that $\mathcal{G}_r(\mathcal{U}_p) = 0$ and $\mathrm{D}\mathcal{G}_r|_{\mathcal{U}_p}$ is invertible along directions transverse to the orbit of $\boldsymbol{\Delta}$, so that the map

$$\mathcal{H} \ni \mathcal{U}_q \mapsto \mathcal{G}_r(\mathcal{U}_q) \in \mathbb{R}^6$$

has nonsingular derivative at $\mathcal{U}_p$, where $\mathcal{H}$ is a six-dimensional submanifold of $\boldsymbol{U}(M)$ transverse at $\mathcal{U}_p$ to the orbit of $\mathcal{U}_p$ under $\boldsymbol{\Delta}$. Since the norm of the inverse of $\mathrm{D}\mathcal{G}_r|_{\mathcal{U}_p}$ on $T_{\mathcal{U}_p}\mathcal{H}$ must be bounded above by a constant independent of ρ, then the (finite-dimensional) inverse function theorem implies that it is possible to find a neighbouring $\mathcal{U}_{p'} \in \mathcal{H}$ so that $G_\rho(\mathcal{U}_{p'}) \equiv 0$ provided ρ is sufficiently small. Furthermore the distance between $\mathcal{U}_p$ and $\mathcal{U}_{p'}$ as points in $\boldsymbol{U}(M)$ is $\mathcal{O}(\rho)$, which is a consequence of the fact that the equation $G_\rho(\mathcal{U}_{p'}) = 0$ implies $\mathcal{G}_r(\mathcal{U}_{p'}) = \mathcal{O}(\rho)$. As indicated above, this now implies that $\nabla \cdot H(\tilde{\Sigma}_r)$ is constant. Then the divergence theorem implies that it must vanish. $\qquad\square$

References

1. Abreu, M.: Kähler geometry of toric manifolds in symplectic coordinates. In: Eliashberg, Y., Khesin, B., Lalonde, F. (eds.) Symplectic and Contact Topology: Interactions and Perspectives (Toronto, ON/Montreal, QC, 2001). Fields Institute Communications, vol. 35, pp. 1–24. American Mathematical Society, Providence, RI (2003)
2. Griffiths, P., Harris, J.: Principles of Algebraic Geometry. Wiley, New York (1978)
3. Hélein, F., Romon, P.: Hamiltonian stationary Lagrangian surfaces in Hermitian symmetric spaces. In: Guest, M., Miyaoka, R., Ohnita, Y. (eds.) Differential Geometry and Integrable Systems (Tokyo, 2000). Contemporary Mathematics, vol. 308, pp. 161–178. American Mathematical Society, Providence, RI (2002)
4. Hélein, F., Romon, P.: Hamiltonian stationary Lagrangian surfaces in $\mathbf{C}^2$. Commun. Anal. Geom. **10**(1), 79–126 (2002)
5. Hélein, F., Romon, P.: Hamiltonian stationary tori in the complex projective plane. Proc. Lond. Math. Soc. (3) **90**(2), 472–496 (2005)
6. Joyce, D.: Lectures on special Lagrangian geometry. In: Hoffman, D. (ed.) Global Theory of Minimal Surfaces. Clay Mathematics Proceedings, vol. 2, pp. 667–695. American Mathematical Society, Providence, RI (2005)

7. Joyce, D., Lee, Y.-I., Schoen, R.: On the existence of Hamiltonian stationary Lagrangian submanifolds in sympectic manifolds. Am. J. Math. **133**(4), 1067–1092 (2011)
8. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, vol. II. Wiley, New York (1996)
9. Lee, Y.-I.: The existence of Hamiltonian stationary tori in Kahler manifolds of any dimension. Preprint: arXiv:10013861v1 [math.DG].
10. Oh, Y.-G.: Second variation and stabilities of minimal Lagrangian submanifolds in Kähler manifolds. Invent. Math. **101**(2), 501–519 (1990)
11. Oh, Y.-G.: Volume minimization of Lagrangian submanifolds under Hamiltonian deformations. Math. Z. **212**(2), 175–192 (1993)
12. Pacard, F., Xu, X.: Constant mean curvature spheres in Riemannian manifolds. Manuscr. Math. **128**(3), 275–295 (2009)
13. Schoen, R.: Special Lagrangian submanifolds. In: Hoffman, D. (ed.) Global Theory of Minimal Surfaces. Clay Mathematics Proceedings, vol. 2, pp. 655–666. American Mathematical Society, Providence, RI (2005)
14. Schoen, R., Wolfson, J.: Minimizing volume among Lagrangian submanifolds. In: Giaquinta, M., Shatah, J., Varadhan, S.R.S. (eds.) Differential Equations: La Pietra 1996. Proceedings of Symposia in Pure Mathematics, vol. 65, pp. 181–199. American Mathematical Society, Providence, RI (1999)
15. Schoen, R., Wolfson, J.: Minimizing area among Lagrangian surfaces: the mapping problem. J. Differ. Geom. **58**(1), 1–86 (2001)
16. Ye, R.: Foliation by constant mean curvature spheres. Pac. J. Math. **47**(2), 381–396 (1991)